

# INFINITE-DIMENSIONAL FROBENIUS MANIFOLDS UNDERLYING THE TODA LATTICE HIERARCHY

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**ABSTRACT.** Following the approach of [Carlet et al, Math. Ann. 349 (2011), 75–115], we construct a class of infinite-dimensional Frobenius manifolds underlying the Toda lattice hierarchy, which are defined on the space of pairs of meromorphic functions with possibly higher-order poles at the origin and at infinity. Moreover, these infinite-dimensional manifolds are shown to contain Frobenius submanifolds of finite dimension that coincide with those on the orbit space of extended affine Weyl groups of type A defined by Dubrovin and Zhang.

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## 1. INTRODUCTION

The concept of Frobenius manifold was introduced by Dubrovin [12, 13] as a geometric formalism of the Witten-Dijkgraaf-E. Virlinde-H. Virlinde (WDVV) equation in topological field theory [10, 23]. Associated to every Frobenius manifold, there is a so-called principal hierarchy of Hamiltonian equations of hydrodynamic type, in which the unknown functions depend on one scalar spatial variable and some time variables. Under certain general assumptions, the principal hierarchy can be extended to a full hierarchy [18, 15] with a tau function that plays an important role in relevant research areas such like topological field theory.

Trials to extend the above programme to integrable systems with two spatial dimensions started in recent years. The first step was made by Carlet, Dubrovin and Mertens [7] in consideration of the dispersionless (2D) Toda lattice hierarchy [22], in which the first nontrivial  $2 + 1$  evolutionary equation reads

$$\partial_t^2 u = \partial_x^2 e^u + \partial_y^2 u \quad (1.1)$$

for unknown function  $u = u(x, y, t)$ . It turns out that the underlying Frobenius manifold is of infinite dimension. Following their approach, a class of infinite-dimensional Frobenius manifolds was constructed by Xu and one of the authors [25] for the dispersionless two-component BKP hierarchy, and these manifolds contain finite-dimensional Frobenius submanifolds corresponding to Coxeter groups of types B and D [1, 13, 26]. Along the same line, now we want to revise and generalize the construction in [7] with some skills developed in [25]. More exactly, for the dispersionless Toda lattice hierarchy, we will find a family of infinite-dimensional Frobenius manifolds labeled by a pair of positive integers  $(M, N)$ , among which the  $(1, 1)$  case is similar, but not exactly the same, with the Frobenius manifold given in [7]. Moreover, these infinite-dimensional manifolds will be shown to contain Frobenius submanifolds of  $M + N$  dimensions that were constructed on the orbit space of extended affine Weyl group  $\tilde{W}^{(N)}(A_{M+N-1})$  [17]. Thus we have more examples to support a normal form for infinite-dimensional Frobenius manifolds and their natural connection to Frobenius manifolds of finite dimensions.

We remark that, Raimondo [19], with a method very different from that mentioned above, proposed an infinite-dimensional Frobenius manifold for the dispersionless Kadomtsev-Petviashvili (KP) equation by using the theory of Schwartz functions. In a recent paper [20], Szablikowski proposed a scheme for constructing Frobenius manifolds of finite or infinite dimensions based on the Rota-Baxter identity and a counterpart of the modified Yang-Baxter equation for classical  $r$ -matrix (see references therein). He suggested an infinite-dimensional Frobenius manifold structure on the space of Laurent series where the dispersionless KP hierarchy is defined, as well as hints to study the case of space consisting of two component of Laurent series to define the dispersionless Toda lattice hierarchy. The relation between the methods in [19, 20] and the

approaches mentioned previously to infinite-dimensional Frobenius manifolds is not clear yet.

Recall that a *Frobenius algebra*  $(A, e, <, >)$  is a commutative associative algebra  $A$  with a unity  $e$  and a non-degenerate symmetric invariant bilinear form (inner product)  $<, >$ . A manifold  $M$  is called a *Frobenius manifold* if on each tangent space  $T_t M$  a Frobenius algebra  $(T_t M, e, <, >)$  is defined depending smoothly on  $t \in M$ , and the following conditions are satisfied:

- (F1) the inner product  $<, >$  is a flat metric on  $M$ , and, with  $\nabla$  being the Levi-Civita connection for this metric, the unity vector field  $e$  satisfies  $\nabla e = 0$ ;
- (F2) let  $c$  be the 3-tensor  $c(X, Y, Z) := < X \cdot Y, Z >$ , then the 4-tensor  $(\nabla_W c)(X, Y, Z)$  is symmetric in the vector fields  $X, Y, Z$  and  $W$ ;
- (F3) there is a so-called Euler vector field  $E$  on  $M$  such that  $\nabla \nabla E = 0$ , and

$$[E, X \cdot Y] - [E, X] \cdot Y - X \cdot [E, Y] = X \cdot Y,$$

$$Lie_E < X, Y > - < [E, X], Y > - < X, [E, Y] > = (2 - d) < X, Y >,$$

where  $d$  is a constant named as the charge of  $M$ .

On an  $n$ -dimensional Frobenius manifold  $M$ , one can choose a system of flat coordinates  $t = (t^1, \dots, t^n)$  such that the unity vector field is  $e = \partial/\partial t^1$ . Note that

$$(\eta_{\alpha\beta})_{n \times n} = \left( < \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} > \right)_{n \times n} \quad (1.2)$$

is a constant non-degenerate matrix, whose inverse is denoted as  $(\eta^{\alpha\beta})_{n \times n}$ . Let

$$c_{\alpha\beta\gamma} = c \left( \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}, \frac{\partial}{\partial t^\gamma} \right), \quad (1.3)$$

then the product of the Frobenius algebra  $T_t M$  reads

$$\frac{\partial}{\partial t^\alpha} \cdot \frac{\partial}{\partial t^\beta} = c_{\alpha\beta}^\gamma \frac{\partial}{\partial t^\gamma}, \quad c_{\alpha\beta}^\gamma = \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}. \quad (1.4)$$

Here and below the convention of summation over repeated Greek indices is assumed.

The structure constants of the Frobenius algebra  $T_t M$  satisfy

$$c_{1\alpha}^\beta = \delta_\alpha^\beta, \quad c_{\alpha\beta}^\epsilon c_{\epsilon\gamma}^\sigma = c_{\alpha\gamma}^\epsilon c_{\epsilon\beta}^\sigma. \quad (1.5)$$

Moreover, there locally exists a smooth function  $F(t)$ , named as *potential* of the Frobenius manifold, such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \quad (1.6)$$

$$Lie_E F = (3 - d)F + \text{quadratic terms in } t. \quad (1.7)$$

In other words, the function  $F$  solves the WDVV equation (1.5)–(1.7), and its third-order derivatives  $c_{\alpha\beta\gamma}$  are called the 3-point correlator functions in topological field theory.

Conversely, given a solution  $F$  of (1.5)–(1.7) (including a flat metric, a unity and a Euler vector field), one can recover the structure of a Frobenius manifold.

A Frobenius manifold  $M$  is said to be *semisimple* if the Frobenius algebras  $T_t M$  are semisimple for generic points  $t \in M$ .

On the Frobenius manifold  $M$ , its cotangent space  $T_t^* M$  also carries a Frobenius algebra structure, with an invariant bilinear form  $\langle dt^\alpha, dt^\beta \rangle^* = \eta^{\alpha\beta}$  and a product given by

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta} dt^\gamma, \quad c_\gamma^{\alpha\beta} = \eta^{\alpha\epsilon} c_{\epsilon\gamma}^\beta. \quad (1.8)$$

Let

$$g^{\alpha\beta} = i_E(dt^\alpha \cdot dt^\beta), \quad (1.9)$$

then  $(dt^\alpha, dt^\beta)^* = g^{\alpha\beta}$  defines a symmetric bilinear form, called the *intersection form*, on  $T_t^* M$ .

The above two bilinear forms on  $T^* M$  compose a pencil  $g^{\alpha\beta} + s \eta^{\alpha\beta}$  of flat metrics with parameter  $s$ , hence they induces a bi-hamiltonian structure  $\{ , \}_2 + s \{ , \}_1$  of hydrodynamic type [16] on the loop space  $\{S^1 \rightarrow M\}$ . Furthermore, on the loop space one can choose a family of functions  $\theta_{\alpha,p}(t)$  with  $\alpha = 1, 2, \dots, n$  and  $p \geq 0$  such that

$$\theta_{\alpha,0} = \eta_{\alpha\beta} t^\beta, \quad \theta_{\alpha,1} = \frac{\partial F}{\partial t^\alpha}, \quad \frac{\partial^2 \theta_{\alpha,p}}{\partial t^\lambda \partial t^\mu} = c_{\lambda\mu}^\epsilon \frac{\partial \theta_{\alpha,p-1}}{\partial t^\epsilon} \text{ for } p > 1. \quad (1.10)$$

The *principal hierarchy* associated to  $M$  is the following system of Hamiltonian equations

$$\frac{\partial t^\gamma}{\partial T^{\alpha,p}} = \left\{ t^\gamma(x), \int_1 \theta_{\alpha,p} dx \right\}, \quad \alpha, \gamma = 1, 2, \dots, n; \quad p \geq 0, \quad (1.11)$$

in which  $x$  is the coordinate of  $S^1$ . This hierarchy can be written in a bi-hamiltonian recursion form that is consistent with (1.10) if certain nonresonant condition is fulfilled.

As typical examples, the orbit space of each (extended) Weyl group is endowed a semisimple Frobenius manifold structure [13, 17]. In Particular, the Frobenius manifolds for Weyl groups have principal hierarchies that coincide with the dispersionless limit of Drinfeld-Sokolov hierarchies [11, 18, 15]. For Frobenius manifolds corresponding to extended affine Weyl groups [17], so far as we know, only in the case of type A their principal hierarchies are clear, which are the dispersionless limit of the extended bigraded Toda hierarchies [6, 8].

The picture for infinite-dimensional Frobenius manifold is similar [7, 19, 25], though examples are much less by now. For instance, Carlet, Dubrovin and Mertens' Frobenius manifold in [7] is associated with an extension of the dispersionless Toda lattice hierarchy [9]. In consideration of the relation between the Toda lattice and the extended bigraded Toda hierarchies, it is natural to study the connection of infinite-dimensional Frobenius manifolds and the finite-dimensional ones for extended affine Weyl groups of type A, as hinted by [25].

Such a connection will be clarified in a revision of the construction of [7], see below.

Let us state the main results of the present paper, with similar notations as in [7, 25].

Given  $0 < \epsilon < 1$ , one has two sets of holomorphic functions on disks of  $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$  as follows:

$$\begin{aligned}\mathcal{H}^- &= \left\{ f(z) = \sum_{i \geq 0} f_i z^{-i} \mid f \text{ holomorphic on } |z| > 1 - \epsilon \right\}, \\ \mathcal{H}^+ &= \left\{ \hat{f}(z) = \sum_{i \geq 0} \hat{f}_i z^i \mid \hat{f} \text{ holomorphic on } |z| < 1 + \epsilon \right\}.\end{aligned}$$

We fix two arbitrary positive integers  $M$  and  $N$ , and consider the following coset

$$\tilde{\mathcal{M}}_{M,N} = (z^N, 0) + z^{N-1} \mathcal{H}^- \times z^{-M} \mathcal{H}^+. \quad (1.12)$$

Any element of this coset is written as  $\mathbf{a}(z) = (a(z), \hat{a}(z))$ , where

$$a(z) = z^N + \sum_{i \leq N-1} v_i z^i, \quad \hat{a}(z) = \sum_{j \geq -M} \hat{v}_j z^j. \quad (1.13)$$

With coordinates given by the coefficients  $v_i$  and  $\hat{v}_j$  in the above expansions, the coset  $\tilde{\mathcal{M}}_{M,N}$  can be considered as an infinite-dimensional manifold.

For  $\mathbf{a}(z) = (a(z), \hat{a}(z)) \in \tilde{\mathcal{M}}_{M,N}$ , we introduce

$$\zeta(z) = a(z) - \hat{a}(z), \quad l(z) = a(z)_{>0} + \hat{a}(z)_{\leq 0}, \quad (1.14)$$

where the subscripts “ $> 0$ ” and “ $\leq 0$ ” mean the projections of a Laurent series to its positive part and nonpositive part respectively.

**Definition 1.1.** Let  $\mathcal{M}_{M,N}$  be a submanifold of  $\tilde{\mathcal{M}}_{M,N}$  consisting of points  $(a(z), \hat{a}(z))$  such that the following conditions are satisfied:

- (M1) the function  $\hat{a}(z)$  has a pole of order  $M$  at 0, namely,  $\hat{v}_{-M} \neq 0$ ;
- (M2) at  $|z| = 1$ ,

$$a(z)\hat{a}'(z) - a'(z)\hat{a}(z) \neq 0, \quad \zeta'(z) \neq 0, \quad l'(z) \neq 0; \quad (1.15)$$

- (M3) the function  $\zeta(z)$  has winding number 1 around 0, which defines a bi-holomorphic map from  $S^1$  to a simple smooth curve  $\Gamma$  around  $\zeta = 0$ .

On  $\mathcal{M}_{M,N}$  we introduce variables

$$\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}} = \{t^i\}_{i \in \mathbb{Z}} \cup \{h^j\}_{j=1}^N \cup \{\hat{h}^k\}_{k=0}^M \quad (1.16)$$

by

$$t^i = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\zeta(z)^{-i}}{i} \frac{dz}{z}, \quad i \in \mathbb{Z} \setminus \{0\}; \quad t^0 = \frac{1}{2\pi i} \oint_{|z|=1} \log \frac{z}{\zeta(z)} \frac{dz}{z}; \quad (1.17)$$

$$h^j = -\frac{N}{j} \text{res}_{z=\infty} l(z)^{j/N} \frac{dz}{z}, \quad j = 1, \dots, N-1; \quad (1.18)$$

$$\hat{h}^0 = \log \hat{v}_{-M}; \quad \hat{h}^k = \frac{M}{k} \text{res}_{z=0} l(z)^{k/M} \frac{dz}{z}, \quad k = 1, \dots, M. \quad (1.19)$$

**Main Theorem.** *For any two positive integers  $M$  and  $N$ , the infinite-dimensional manifold  $\mathcal{M}_{M,N}$  is a semisimple Frobenius manifold with a system of flat coordinates (1.16) such that*

(i) *the unity vector field is*

$$\mathbf{e} = \frac{\partial}{\partial \hat{h}^M}; \quad (1.20)$$

(ii) *the potential  $\mathcal{F}_{M,N}$  is*

$$\begin{aligned} \mathcal{F}_{M,N} = & \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( \frac{1}{2} \zeta(z_1) \zeta(z_2) - \zeta(z_1) l(z_2) + l(z_1) \zeta(z_2) \right) \times \\ & \times \log \left( \frac{z_2 - z_1}{z_2} \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} - \frac{1}{(2\pi i)^2} \oint_{|z|=1} \left( \frac{1}{2} \zeta(z) + l(z) \right) \frac{dz}{z} \times \\ & \times \oint_{|z|=1} \zeta(z) \left( \log \frac{\zeta(z)}{z} - 1 \right) \frac{dz}{z} + F_{M,N}, \end{aligned} \quad (1.21)$$

where  $F_{M,N}$  is a function of  $\mathbf{h} \cup \hat{\mathbf{h}}$  determined by

$$\frac{\partial^3 F_{M,N}}{\partial u \partial v \partial w} = -(\text{res}_{z=\infty} + \text{res}_{z=0}) \frac{\partial_u l(z) \cdot \partial_v l(z) \cdot \partial_w l(z)}{z^2 l'(z)} dz \quad (1.22)$$

for any  $u, v, w \in \mathbf{h} \cup \hat{\mathbf{h}}$ ;

(iii) *the Euler vector field is*

$$\begin{aligned} \mathcal{E}_{M,N} = & - \sum_{i \in \mathbb{Z}} i t^i \frac{\partial}{\partial t^i} - \frac{N-1}{N} \frac{\partial}{\partial t^0} + \sum_{j=1}^{N-1} \frac{j}{N} h^j \frac{\partial}{\partial h^j} + \\ & + \sum_{k=1}^M \frac{k}{M} \hat{h}^k \frac{\partial}{\partial \hat{h}^k} + \frac{N+M}{N} \frac{\partial}{\partial \hat{h}^0}, \end{aligned} \quad (1.23)$$

and the charge of the Frobenius manifold is  $d = 1$ .

This theorem will be proved in Section 2. There we will also write down the flat metric and the intersection form for this semisimple Frobenius manifold. Observe that the infinite-dimensional Frobenius manifold  $\mathcal{M}_{1,1}$  is similar, but not the same, with the one constructed in [7] (the difference occurs from the definition of flat metric, see (2.11) below).

One can see that  $F_{M,N}$ , depending polynomially on  $\mathbf{h} \cup \hat{\mathbf{h}} \cup \{e^{\hat{h}^0}\}$ , is indeed the potential for the semisimple Frobenius manifold, say,  $M(\tilde{A}_{M+N-1}; N)$ , on the orbit space of the extended Weyl group  $\tilde{W}^{(N)}(A_{M+N-1})$ , see [17]. Hence we have

**Corollary 1.2.** *The infinite-dimensional Frobenius manifold  $\mathcal{M}_{M,N}$  contains  $M(\tilde{A}_{M+N-1}; N)$  as its semisimple Frobenius submanifold of dimension  $M+N$ .*

In Section 3 we will show that the flat pencil on  $T_{\mathbf{a}}^*\mathcal{M}_{M,N}$  coincides with the one that induces the dispersionless limit of the bi-hamiltonian structure obtained in [24] for the Toda lattice hierarchy. Accordingly the dispersionless Toda lattice hierarchy is part of the principal hierarchy associated to the Frobenius manifold  $\mathcal{M}_{M,N}$ . The complete principal hierarchy, in contrast to the two-component BKP case [25], cannot be derived by applying only the bi-hamiltonian recursion relation. As indicated by Carlet and Mertens [9], one needs to explicitly solve the flatness equations for the deformed flat connection (the Levi-Civita connection  $\nabla$  deformed by the product of the Euler vector field) on the infinite-dimensional Frobenius manifold. We shall leave this problem open here.

The last section is devoted to the conclusion.

## 2. CONSTRUCTION OF FROBENIUS MANIFOLDS

We start to equip a semisimple Frobenius manifold structure to the infinite-dimensional space

$$\mathcal{M}_{M,N} \subset (z^N, 0) + z^{N-1}\mathcal{H}^- \times z^{-M}\mathcal{H}^+$$

defined by the conditions (M1)–(M3) above with arbitrary positive integers  $M$  and  $N$ .

**2.1. Flat metric.** Recall that every element of  $\mathcal{M}_{M,N}$  has the form

$$\mathbf{a} = (a(z), \hat{a}(z)) = \left( z^N + \sum_{i \leq N-1} v_i z^i, \sum_{j \geq -M} \hat{v}_j z^j \right). \quad (2.1)$$

Let us describe the tangent and the cotangent bundles on  $\mathcal{M}_{M,N}$  with Laurent series. At a point  $\mathbf{a} \in \mathcal{M}_{M,N}$  we identify a vector  $\partial$  in the tangent space with its action  $(\partial a(z), \partial \hat{a}(z))$  on the “point”. Hence the tangent space is identified with a space of pairs of Laurent series as

$$T_{\mathbf{a}}\mathcal{M}_{M,N} = z^{N-1}\mathcal{H}^- \times z^{-M}\mathcal{H}^+. \quad (2.2)$$

Clearly this space has a natural basis given by

$$\frac{\partial}{\partial v_i} = (z^i, 0), \quad i \leq N-1; \quad \frac{\partial}{\partial \hat{v}_j} = (0, z^j), \quad j \geq -M. \quad (2.3)$$

Accordingly, we write the cotangent space as

$$T_{\mathbf{a}}^*\mathcal{M}_{M,N} = z^{-N+1}\mathcal{H}^+ \times z^M\mathcal{H}^-, \quad (2.4)$$

and the pairing of a covector  $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T_{\mathbf{a}}^*\mathcal{M}_{M,N}$  with a vector  $\mathbf{X} = (X(z), \hat{X}(z)) \in T_{\mathbf{a}}\mathcal{M}_{M,N}$  reads

$$\langle \boldsymbol{\omega}, \mathbf{X} \rangle = \frac{1}{2\pi i} \oint_{|z|=1} [\omega(z)X(z) + \hat{\omega}(z)\hat{X}(z)] \frac{dz}{z}. \quad (2.5)$$

Clearly, the cotangent space has a dual basis with respect to (2.3) as follows

$$dv_i = (z^{-i}, 0), \quad i \leq N-1; \quad d\hat{v}_j = (0, z^{-j}), \quad j \geq -M. \quad (2.6)$$

We introduce two generating functions for covectors

$$da(p) := \sum_{i \leq N-1} dv_i p^i = \left( \frac{p^N}{z^{N-1}(p-z)}, 0 \right), \quad |z| < |p|, \quad (2.7)$$

$$d\hat{a}(p) := \sum_{j \geq -M} d\hat{v}_j p^j = \left( 0, \frac{z^{M+1}}{p^M(z-p)} \right), \quad |z| > |p|. \quad (2.8)$$

The Cauchy integral formula implies the following simple but useful lemma.

**Lemma 2.1.** *The following statements for the generating functions (2.7)–(2.8) hold true:*

$$(i) \text{ for any vector } \mathbf{X} = (X(z), \hat{X}(z)) \in T_{\mathbf{a}}\mathcal{M}_{M,N},$$

$$\langle da(p), \mathbf{X} \rangle = X(p), \quad \langle d\hat{a}(p), \mathbf{X} \rangle = \hat{X}(p); \quad (2.9)$$

(ii) for any covector  $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T_{\mathbf{a}}^*\mathcal{M}_{M,N}$ , we have

$$\boldsymbol{\omega} = \frac{1}{2\pi i} \oint_{|p|=1} [\omega(p)da(p) + \hat{\omega}(p)d\hat{a}(p)] \frac{dp}{p}. \quad (2.10)$$

On  $T_{\mathbf{a}}^*\mathcal{M}_{M,N}$ , we define a symmetric bilinear form by

$$\langle d\alpha(p), d\beta(q) \rangle^* = \frac{pq}{p-q} (\alpha'(p) - \beta'(q)), \quad (2.11)$$

where  $\alpha'(p) = \partial\alpha(p)/\partial p$  and  $\alpha, \beta \in \{a, \hat{a}\}$ . Observe that even in the case  $M = N = 1$ , this bilinear form is different from the one used in [7] (see Equation (1.16) there).

By using the nondegenerate pairing (2.5), one defines a linear map

$$\eta : T_{\mathbf{a}}^*\mathcal{M}_{M,N} \rightarrow T_{\mathbf{a}}\mathcal{M}_{M,N} \quad (2.12)$$

such that

$$\langle \boldsymbol{\omega}_1, \eta(\boldsymbol{\omega}_2) \rangle = \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle^* \quad (2.13)$$

for any covectors  $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in T_{\mathbf{a}}^*\mathcal{M}_{M,N}$ .

**Lemma 2.2.** *The map  $\eta$  defined in (2.12)–(2.13) can be represented explicitly as*

$$\begin{aligned} \eta(\boldsymbol{\omega})(z) = & \left( za'(z)[\omega(z) + \hat{\omega}(z)]_{<0} - z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{<0}, \right. \\ & \left. - z\hat{a}'(z)[\omega(z) + \hat{\omega}(z)]_{\geq 0} + z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{\geq 0} \right) \end{aligned} \quad (2.14)$$

with arbitrary  $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) \in T_{\mathbf{a}}^*\mathcal{M}_{M,N}$ . Moreover, the linear map  $\eta$  is a bijection.

*Proof.* It follows from (2.9) and (2.13) that

$$\eta(d\beta(q)) = (\langle da(z), d\beta(q) \rangle^*, \langle d\hat{a}(z), d\beta(q) \rangle^*), \quad \beta \in \{a, \hat{a}\}. \quad (2.15)$$

Hence by using (2.10) and  $\langle \boldsymbol{\omega}_1, \eta(\boldsymbol{\omega}_2) \rangle = \langle \eta(\boldsymbol{\omega}_1), \boldsymbol{\omega}_2 \rangle$  we have

$$\eta(\boldsymbol{\omega})(z) = \frac{1}{2\pi i} \oint_{|q|=1} [\omega(q)\eta(da(q)) + \hat{\omega}(q)\eta(d\hat{a}(q))] \frac{dq}{q}$$



$$\begin{aligned}
&= \left( \frac{1}{2\pi i} \oint_{|q|<|z|} \langle da(z), \omega(q)da(q) + \hat{\omega}(q)d\hat{a}(q) \rangle^* \frac{dq}{q}, \right. \\
&\quad \left. \frac{1}{2\pi i} \oint_{|q|>|z|} \langle d\hat{a}(z), \omega(q)da(q) + \hat{\omega}(q)d\hat{a}(q) \rangle^* \frac{dq}{q} \right) \\
&= \left( \frac{1}{2\pi i} \oint_{|q|<|z|} \frac{q}{1-q/z} [a'(z)(\omega(q) + \hat{\omega}(q)) - (\omega(q)a'(q) + \hat{\omega}(q)\hat{a}'(q))] \frac{dq}{q}, \right. \\
&\quad \left. - \frac{1}{2\pi i} \oint_{|q|>|z|} \frac{z}{1-z/q} [\hat{a}'(z)(\omega(q) + \hat{\omega}(q)) - (\omega(q)a'(q) + \hat{\omega}(q)\hat{a}'(q))] \frac{dq}{q} \right) \\
&= \left( za'(z)[\omega(z) + \hat{\omega}(z)]_{<0} - z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{<0}, \right. \\
&\quad \left. -z\hat{a}'(z)[\omega(z) + \hat{\omega}(z)]_{\geq 0} + z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{\geq 0} \right). \tag{2.16}
\end{aligned}$$

This is just (2.14). Moreover, it implies the map  $\eta$  being surjective.

Next we want to show that the map  $\eta$  is invertible. Suppose that

$$\begin{aligned}
\mathbf{X} = (X(z), \hat{X}(z)) &= \left( \sum_{i \leq N-1} X_i z^i, \sum_{j \geq -M} \hat{X}_j z^j \right) \in T_{\mathbf{a}} \mathcal{M}_{M,N}, \\
\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z)) &= \left( \sum_{i \geq -N+1} \omega_i z^i, \sum_{j \leq M} \hat{\omega}_j z^j \right) \in T_{\mathbf{a}}^* \mathcal{M}_{M,N}
\end{aligned}$$

satisfy  $\mathbf{X} = \eta(\boldsymbol{\omega})$ , namely,

$$X(z) = za'(z)[\omega(z) + \hat{\omega}(z)]_{<0} - z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{<0}, \tag{2.17}$$

$$\hat{X}(z) = -z\hat{a}'(z)[\omega(z) + \hat{\omega}(z)]_{\geq 0} + z[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{\geq 0}. \tag{2.18}$$

It follows that

$$X(z) - \hat{X}(z) = z(a'(z) - \hat{a}'(z))(\hat{\omega}(z)_{<0} - \omega(z)_{\geq 0}). \tag{2.19}$$

Hence we obtain

$$\omega(z)_{\geq 0} = - \left( \frac{X(z) - \hat{X}(z)}{z(a'(z) - \hat{a}'(z))} \right)_{\geq 0}, \quad \hat{\omega}(z)_{<0} = \left( \frac{X(z) - \hat{X}(z)}{z(a'(z) - \hat{a}'(z))} \right)_{<0}. \tag{2.20}$$

Here we have used the property  $\zeta'(z) \neq 0$  in the definition of  $\mathcal{M}_{M,N}$ .

On the one hand, by using (2.17) and (2.18) one has

$$X(z)_{>0} = (za'(z)[\omega(z) + \hat{\omega}(z)]_{<0})_{>0} = (za'(z)_{>0}[\omega(z) + \hat{\omega}(z)]_{<0})_{>0}, \tag{2.21}$$

$$\hat{X}(z)_{\leq 0} = -(z\hat{a}'(z)[\omega(z) + \hat{\omega}(z)]_{\geq 0})_{\leq 0} = -(z\hat{a}'(z)_{<0}[\omega(z) + \hat{\omega}(z)]_{\geq 0})_{\leq 0}. \tag{2.22}$$

Writing

$$\omega(z) + \hat{\omega}(z) = \sum_{k \in \mathbb{Z}} \tilde{\omega}_k z^k,$$

then the equations (2.21)–(2.22) can be rewritten as

$$\begin{pmatrix} X_{N-1} \\ \vdots \\ X_1 \end{pmatrix} = K_{N-1} \begin{pmatrix} \tilde{\omega}_{-1} \\ \vdots \\ \tilde{\omega}_{-N+1} \end{pmatrix}, \quad \begin{pmatrix} \hat{X}_{-M} \\ \vdots \\ \hat{X}_0 \end{pmatrix} = \hat{K}_{M+1} \begin{pmatrix} \tilde{\omega}_0 \\ \vdots \\ \tilde{\omega}_M \end{pmatrix}, \quad (2.23)$$

where

$$K_{N-1} = \begin{pmatrix} N & & & & \\ (N-1)v_{N-1} & N & & & \\ (N-2)v_{N-2} & (N-1)v_{N-1} & N & & \\ \vdots & \ddots & \ddots & \ddots & \\ 2v_2 & \cdots & (N-2)v_{N-2} & (N-1)v_{N-1} & N \end{pmatrix} \quad (2.24)$$

and

$$\hat{K}_{M+1} = \begin{pmatrix} M\hat{v}_{-M} & & & & \\ (M-1)\hat{v}_{-M+1} & M\hat{v}_{-M} & & & \\ \vdots & \ddots & \ddots & & \\ \hat{v}_{-1} & & \ddots & M\hat{v}_{-M} & \\ 0 & \hat{v}_{-1} & \cdots & (M-1)\hat{v}_{-M+1} & M\hat{v}_{-M} \end{pmatrix}. \quad (2.25)$$

Both matrices  $K_{N-1}$  and  $\hat{K}_{M+1}$  are nondegenerate, hence  $\tilde{\omega}_{-N+1}, \dots, \tilde{\omega}_M$  can be solved from (2.21)–(2.22). This fact together with (2.20) leads to that  $\boldsymbol{\omega} = (\omega(z), \hat{\omega}(z))$  is uniquely determined by  $\mathbf{X} = (X(z), \hat{X}(z))$ . Therefore the lemma is proved.  $\square$

With the help of the bijection  $\eta$ , the bilinear form (2.11) on the cotangent space induces a symmetric bilinear form on  $T_{\mathbf{a}}\mathcal{M}_{M,N}$  as

$$\langle \partial_1, \partial_2 \rangle := \langle \eta^{-1}(\partial_1), \partial_2 \rangle = \langle \eta^{-1}(\partial_1), \eta^{-1}(\partial_2) \rangle^*. \quad (2.26)$$

Recall the property (1.15) for the functions

$$\zeta(z) = a(z) - \hat{a}(z), \quad l(z) = a(z)_{>0} + \hat{a}(z)_{\leq 0}. \quad (2.27)$$

One has the following lemma.

**Lemma 2.3.** *The bilinear form (2.26) can be represented as follows: for any tangent vectors  $\partial_1, \partial_2 \in T_{\mathbf{a}}\mathcal{M}_{M,N}$ ,*

$$\begin{aligned} \langle \partial_1, \partial_2 \rangle = & -\frac{1}{2\pi i} \oint_{|z|=1} \frac{\partial_1 \zeta(z) \cdot \partial_2 \zeta(z)}{z^2 \zeta'(z)} dz \\ & - \operatorname{res}_{z=\infty} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{z^2 l'(z)} dz - \operatorname{res}_{z=0} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{z^2 l'(z)} dz. \end{aligned} \quad (2.28)$$

*Proof.* Assume  $\eta^{-1}(\partial_1) = (\omega(z), \hat{\omega}(z))$ , then we have

$$\langle \eta^{-1}(\partial_1), \partial_2 \rangle = \frac{1}{2\pi i} \oint_{|z|=1} [\omega(z) \partial_2 a(z) + \hat{\omega}(z) \partial_2 \hat{a}(z)] \frac{dz}{z}$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \oint_{|z|=1} [(\omega(z)_{\geq 0} - \hat{\omega}(z)_{<0})(\partial_2 a(z) - \partial_2 \hat{a}(z))] \frac{dz}{z} \\
&\quad + \frac{1}{2\pi i} \oint_{|z|=1} [\omega(z) + \hat{\omega}(z)]_{<0} \partial_2 a(z) \frac{dz}{z} \\
&\quad + \frac{1}{2\pi i} \oint_{|z|=1} [\omega(z) + \hat{\omega}(z)]_{\geq 0} \partial_2 \hat{a}(z) \frac{dz}{z}. \tag{2.29}
\end{aligned}$$

The three integrals on the right hand side are denoted as  $I_1$ ,  $I_2$  and  $I_3$ , respectively. Let us calculate them separately.

Firstly, by using (2.27) and  $\partial_1 = (\partial_1 a(z), \partial_1 \hat{a}(z))$ , the formulae (2.20) read

$$\omega(z)_{\geq 0} = - \left( \frac{\partial_1 \zeta(z)}{z \zeta'(z)} \right)_{\geq 0}, \quad \hat{\omega}(z)_{<0} = \left( \frac{\partial_1 \zeta(z)}{z \zeta'(z)} \right)_{<0}.$$

Hence we have

$$I_1 = -\frac{1}{2\pi i} \oint_{|z|=1} \frac{\partial_1 \zeta(z) \cdot \partial_2 \zeta(z)}{z^2 \zeta'(z)} dz.$$

Secondly, recall

$$l'(z) = Nz^{N-1} + \sum_{i=1}^{N-1} i v_i z^{i-1} - \sum_{j=1}^M j \hat{v}_{-j} z^{-j-1}.$$

For simplicity, we denote  $\Lambda(n) = (\delta_{i,j+1})_{n \times n}$  for positive integer  $n$  and  $\Lambda(0) = 0$ . Observe that the matrix (2.24) is just

$$K_{N-1} = \left( \frac{l'(z)}{z^{N-1}} \right)_{z^{-1} \rightarrow \Lambda(N-1)}. \tag{2.30}$$

Let's take the following expansion near  $z = \infty$

$$\frac{1}{l'(z)} = \frac{1}{z^{N-1}} (f_0 + f_{-1} z^{-1} + f_{-2} z^{-2} + \cdots),$$

and set  $f_i = 0$  for  $i > 0$ , then we have

$$K_{N-1}^{-1} = \left( \frac{z^{N-1}}{l'(z)} \right)_{z^{-1} \rightarrow \Lambda(N-1)} = (f_{-i+j})_{(N-1) \times (N-1)}. \tag{2.31}$$

Thus by using the first relation in (2.23), we know

$$\begin{aligned}
I_2 &= (\partial_1 v_1, \dots, \partial_1 v_{N-1}) K_{N-1}^{-1} \begin{pmatrix} \partial_2 v_{N-1} \\ \vdots \\ \partial_2 v_1 \end{pmatrix} \\
&= \sum_{i,j=1}^{N-1} \partial_1 v_i \cdot f_{-i+j} \cdot \partial_2 v_{N-j} \\
&= -\text{res}_{z=\infty} \left( \sum_{i,j=1}^{N-1} \partial_1 v_i z^i \cdot f_{-i+j} z^{-i+j} \cdot \partial_2 v_{N-j} z^{N-j} \cdot z^{-N-1} \right) dz
\end{aligned}$$

$$= -\text{res}_{z=\infty} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{z^2 l'(z)} dz.$$

Thirdly, in the same way as above one has

$$\hat{K}_{M+1} = - \left( l'(z) z^{M+1} \right)_{z \rightarrow \Lambda(M+1)} \quad (2.32)$$

and

$$\hat{K}_{M+1}^{-1} = - \left( \frac{1}{l'(z) z^{M+1}} \right)_{z \rightarrow \Lambda(M+1)} = -(g_{i-j})_{(M+1) \times (M+1)}, \quad (2.33)$$

where  $g_m$  are given by the following expansion near  $z = 0$ :

$$\frac{1}{l'(z)} = z^{M+1} (g_0 + g_1 z + g_2 z^2 + \cdots),$$

and  $g_m = 0$  for  $m < 0$ . The second equation in (2.23) leads to

$$\begin{aligned} I_3 &= (\partial_1 \hat{v}_0, \dots, \partial_1 \hat{v}_{-M}) \hat{K}_{M+1}^{-1} \begin{pmatrix} \partial_2 \hat{v}_{-M} \\ \vdots \\ \partial_2 \hat{v}_0 \end{pmatrix} \\ &= - \sum_{i,j=0}^M \partial_1 \hat{v}_{-i} \cdot g_{i-j} \cdot \partial_2 \hat{v}_{j-M} \\ &= -\text{res}_{z=0} \left( \sum_{i,j=0}^M \partial_1 \hat{v}_{-i} z^{-i} \cdot g_{i-j} z^{i-j} \cdot \partial_2 \hat{v}_{j-M} z^{j-M} \cdot z^{M-1} \right) dz \\ &= -\text{res}_{z=0} \frac{\partial_1 l(z) \cdot \partial_2 l(z)}{z^2 l'(z)} dz. \end{aligned}$$

Thus we complete the proof of this lemma.  $\square$

Next we want to choose a system of “flat” coordinates for the bilinear form (2.28). According to the definition of  $\mathcal{M}_{M,N}$ , one can consider the inverse function of  $\zeta(z)$  such that

$$z = z(\zeta) : \Gamma \longrightarrow S^1, \quad (2.34)$$

and this function is holomorphic on a neighborhood of the curve  $\Gamma$  surrounding  $\zeta = 0$ . We assume the following Riemann-Hilbert factorization on the  $\zeta$ -plane

$$z(\zeta) = f_0^{-1}(\zeta) f_\infty(\zeta) \quad \text{for } \zeta \in \Gamma, \quad (2.35)$$

where the functions  $f_0(\zeta)$  and  $f_\infty(\zeta)/\zeta$  are holomorphic inside and outside the curve  $\Gamma$ , respectively. Let us fix this factorization by normalizing

$$f_\infty(\zeta) = \zeta + O(1), \quad |\zeta| \rightarrow \infty.$$

Now we have Taylor expansions with coefficients  $t^i$ :

$$\log f_0(\zeta) = - \sum_{i \geq 0} t^i \zeta^i, \quad |\zeta| \rightarrow 0; \quad (2.36)$$

$$\log \frac{f_\infty(\zeta)}{\zeta} = \sum_{i \geq 1} \frac{t^{-i}}{\zeta^i}, \quad |\zeta| \rightarrow \infty. \quad (2.37)$$

In other words,

$$t^i = \frac{1}{2\pi i} \oint_{\Gamma} \zeta^{-i-1} \log \frac{z(\zeta)}{\zeta} d\zeta. \quad (2.38)$$

On the other hand, recall  $l(z) = z^N + \sum_{i=1}^{N-1} v_i z^i + \sum_{j=0}^M \hat{v}_{-j} z^{-j}$  and denote

$$\chi(z) := l(z)^{1/N} \text{ near } \infty, \quad \hat{\chi}(z) := l(z)^{1/M} \text{ near } 0. \quad (2.39)$$

Let  $z(\chi)$  and  $z(\hat{\chi})$  be the inverse functions of  $\chi(z)$  and  $\hat{\chi}(z)$  respectively. Their logarithms can be expanded as follows:

$$\log z(\chi) = \log \chi - \frac{1}{N} (h^1 \chi^{-1} + \dots + h^{N-1} \chi^{-N+1}) + O(\chi^{-N}), \quad |\chi| \rightarrow \infty; \quad (2.40)$$

$$\log z(\hat{\chi}) = -\log \hat{\chi} + \frac{1}{M} (\hat{h}^0 + \hat{h}^1 \hat{\chi}^{-1} + \dots + \hat{h}^M \hat{\chi}^{-M}) + O(\chi^{-M-1}), \quad |\hat{\chi}| \rightarrow \infty, \quad (2.41)$$

where in particular  $\hat{h}^0 = \log \hat{v}_{-M}$ .

Observe that the variables

$$\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}} = \{t^i \mid i \in \mathbb{Z}\} \cup \{h^j \mid j = 1, \dots, N-1\} \cup \{\hat{h}^k \mid k = 0, \dots, M\}, \quad (2.42)$$

determine  $\zeta(z)$  and  $l(z) = \chi(z)^N = \hat{\chi}(z)^M$  uniquely, hence also

$$a(z) = l(z) + \zeta(z)_{\leq 0}, \quad \hat{a}(z) = l(z) - \zeta(z)_{> 0}. \quad (2.43)$$

It means that (2.42) is indeed a system of coordinates on the manifold  $\mathcal{M}_{M,N}$ .

**Proposition 2.4.** *The above coordinates  $\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$  can be defined equivalently by (1.17)–(1.19). The bilinear form  $\langle \cdot, \cdot \rangle$  in (2.28) satisfies*

$$\langle \frac{\partial}{\partial t^{i_1}}, \frac{\partial}{\partial t^{i_2}} \rangle = -\delta_{i_1+i_2, -1}, \quad i_1, i_2 \in \mathbb{Z}; \quad (2.44)$$

$$\langle \frac{\partial}{\partial h^{j_1}}, \frac{\partial}{\partial h^{j_2}} \rangle = \frac{1}{N} \delta_{j_1+j_2, N}, \quad j_1, j_2 \in \{1, 2, \dots, N-1\}; \quad (2.45)$$

$$\langle \frac{\partial}{\partial \hat{h}^{k_1}}, \frac{\partial}{\partial \hat{h}^{k_2}} \rangle = \frac{1}{M} \delta_{k_1+k_2, M}, \quad k_1, k_2 \in \{0, 1, \dots, M\} \quad (2.46)$$

and any other pairing between these vector fields vanishes. Consequently, the bilinear form  $\langle \cdot, \cdot \rangle$  is a nondegenerate flat metric on  $T_{\mathbf{a}} \mathcal{M}_{M,N}$  with flat coordinates  $\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$ .

The proof of this proposition is mainly based on following lemma.

**Lemma 2.5.** *The coordinates  $\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$  satisfy*

$$\frac{\partial \zeta(z)}{\partial t^i} = -z \zeta^i(z) \zeta'(z), \quad (2.47)$$

$$\frac{\partial l(z)}{\partial h^j} = (z \chi(z)^{N-j-1} \chi'(z))_{> 0}, \quad (2.48)$$

$$\frac{\partial l(z)}{\partial \hat{h}^k} = -(z\hat{\chi}(z)^{M-k-1}\hat{\chi}'(z))_{\leq 0}, \quad (2.49)$$

and that the following derivatives vanish:

$$\frac{\partial \zeta(z)}{\partial h^j} = \frac{\partial \zeta(z)}{\partial \hat{h}^k} = \frac{\partial l(z)}{\partial t^i} = \left( \frac{\partial l(z)}{\partial h^j} \right)_{\leq 0} = \left( \frac{\partial l(z)}{\partial \hat{h}^k} \right)_{> 0} = 0. \quad (2.50)$$

*Proof.* By using

$$\zeta^i = \frac{\partial}{\partial t^i} \log \frac{z(\zeta)}{\zeta} = \frac{1}{z(\zeta)} \frac{\partial z(\zeta)}{\partial t^i} \quad (2.51)$$

and the “thermodynamical identity”

$$\left. \frac{\partial \zeta(z)}{\partial t^i} \right|_{z=z(\zeta)} + \zeta'(z)|_{z=z(\zeta)} \frac{\partial z(\zeta)}{\partial t^i} = 0, \quad (2.52)$$

we obtain (2.47). Similarly, by using

$$\begin{aligned} \frac{\partial z(\chi)}{\partial h^j} &= -z(\chi) \left( \frac{1}{N} \chi^{-j} + O(\chi^{-N}) \right), \quad |\chi| \rightarrow \infty; \\ \frac{\partial z(\hat{\chi})}{\partial \hat{h}^k} &= z(\hat{\chi}) \left( \frac{1}{M} \hat{\chi}^{-k} + O(\hat{\chi}^{-M-1}) \right), \quad |\hat{\chi}| \rightarrow \infty \end{aligned}$$

and the corresponding “thermodynamical identities”, we derive

$$\frac{\partial \chi(z)}{\partial h^j} = z\chi'(z) \left( \frac{1}{N} \chi(z)^{-j} + O(z^{-N}) \right), \quad |z| \rightarrow \infty; \quad (2.53)$$

$$\frac{\partial \hat{\chi}(z)}{\partial \hat{h}^k} = -z\hat{\chi}'(z) \left( \frac{1}{M} \hat{\chi}(z)^{-k} + O(z^{M+1}) \right), \quad |z| \rightarrow 0. \quad (2.54)$$

Thus the equalities (2.48)–(2.49) are obtained. The derivatives (2.50) being zero is trivial. Therefore the lemma is proved.  $\square$

*Proof of Proposition 2.4* The first assertion follows from a short calculation. For the second assertion, with  $i_1, i_2 \in \mathbb{Z}$  we have

$$\begin{aligned} \left\langle \frac{\partial}{\partial t^{i_1}}, \frac{\partial}{\partial t^{i_2}} \right\rangle &= -\frac{1}{2\pi i} \oint_{|z|=1} \frac{\frac{\partial \zeta(z)}{\partial t^{i_1}} \cdot \frac{\partial \zeta(z)}{\partial t^{i_2}}}{z^2 \zeta'(z)} dz \\ &= -\frac{1}{2\pi i} \oint_{|z|=1} \zeta^{i_1+i_2}(z) \zeta'(z) dz \\ &= -\frac{1}{2\pi i} \oint_{\Gamma} \zeta^{i_1+i_2} d\zeta = -\delta_{i_1+i_2, -1}. \end{aligned} \quad (2.55)$$

For  $1 \leq j_1, j_2 \leq N-1$ ,

$$\begin{aligned} \left\langle \frac{\partial}{\partial h^{j_1}}, \frac{\partial}{\partial h^{j_2}} \right\rangle &= -\text{res}_{z=\infty} \frac{\frac{\partial l(z)}{\partial h^{j_1}} \cdot \frac{\partial l(z)}{\partial h^{j_2}}}{z^2 l'(z)} dz \\ &= -\text{res}_{z=\infty} \frac{\chi(z)^{N-1}}{N} \left( \chi(z)^{-j_1} + O(z^{-N}) \right) \left( \chi(z)^{-j_2} + O(z^{-N}) \right) \chi'(z) dz \end{aligned}$$

$$= -\text{res}_{\chi=\infty} \frac{\chi^{N-1-j_1-j_2}}{N} d\chi = \frac{1}{N} \delta_{j_1+j_2, N}. \quad (2.56)$$

Finally, for  $0 \leq k_1, k_2 \leq M$ ,

$$\begin{aligned} & \left\langle \frac{\partial}{\partial \hat{h}^{k_1}}, \frac{\partial}{\partial \hat{h}^{k_2}} \right\rangle = -\text{res}_{z=0} \frac{\frac{\partial l(z)}{\partial \hat{h}^{k_1}} \cdot \frac{\partial l(z)}{\partial \hat{h}^{k_2}}}{z^2 l'(z)} dz \\ &= -\text{res}_{z=0} \frac{\hat{\chi}(z)^{M-1}}{M} \left( \hat{\chi}(z)^{-k_1} + O(z^{M+1}) \right) \left( \hat{\chi}(z)^{-k_2} + O(z^{M+1}) \right) \hat{\chi}'(z) dz \\ &= -\text{res}_{\hat{\chi}=\infty} \frac{\hat{\chi}^{M-1-k_1-k_2}}{M} d\hat{\chi} = \frac{1}{M} \delta_{k_1+k_2, M}. \end{aligned}$$

Clearly all other pairings between these vectors vanish. The proposition is proved.  $\square$

**2.2. Frobenius algebra structure.** In order to endow a Frobenius algebra structure on the tangent space of  $\mathcal{M}_{M,N}$ , let us introduce a product on the cotangent space  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$  first:

$$d\alpha(p) \cdot d\beta(q) = \frac{pq}{p-q} \left( \alpha'(p) d\beta(q) - \beta'(q) d\alpha(p) \right) \quad (2.57)$$

with  $\alpha'(p) = \partial\alpha(p)/\partial p$  and  $\alpha, \beta \in \{a, \hat{a}\}$ .

**Lemma 2.6.** *On  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$  the following assertions hold true:*

- (i) *the multiplication defined by (2.57) is associative and commutative; more generally, for  $\alpha_i \in \{a, \hat{a}\}$  one has*

$$d\alpha_1(p_1) \cdot d\alpha_2(p_2) \cdots d\alpha_k(p_k) = \sum_{i=1}^k \left( \prod_{j \neq i} \frac{\alpha'_j(p_j)}{p_i^{-1} - p_j^{-1}} \right) d\alpha_i(p_i). \quad (2.58)$$

- (ii) *the bilinear form  $\langle \cdot, \cdot \rangle^*$  given in (2.11) is invariant with respect to the above multiplication.*

*Proof.* The commutativity of (2.57) is obvious. The formula (2.58) can be verified by an induction, which yields the associativity of the multiplication. Hence we deduce the first assertion.

The second assertion follows from

$$\begin{aligned} \langle d\alpha(p) \cdot d\beta(q), d\gamma(r) \rangle^* &= \frac{\alpha'(p)\beta'(q)}{(p^{-1} - r^{-1})(r^{-1} - q^{-1})} + \frac{\beta'(q)\gamma'(r)}{(q^{-1} - p^{-1})(p^{-1} - r^{-1})} \\ &\quad + \frac{\gamma'(r)\alpha'(p)}{(r^{-1} - q^{-1})(q^{-1} - p^{-1})} \\ &= \langle d\beta(q) \cdot d\gamma(r), d\alpha(p) \rangle^* \end{aligned}$$

with  $\alpha, \beta, \gamma \in \{a, \hat{a}\}$ .  $\square$

**Lemma 2.7.** *The multiplication (2.57) can be represented in Laurent series as*

$$\omega_1 \cdot \omega_2 = z \left( [\omega_2(z)(\omega_1(z)a'(z))]_{\geq 0} - \omega_2(z)(\hat{\omega}_1(z)\hat{a}'(z))_{< 0} \right)$$

$$\begin{aligned}
& -\omega_1(z)(\omega_2(z)a'(z))_{<0} - \omega_1(z)(\hat{\omega}_2(z)\hat{a}'(z))_{<0}]_{\geq -N}, \\
& [\hat{\omega}_2(z)(\omega_1(z)a'(z))_{\geq 0} + \hat{\omega}_2(z)(\hat{\omega}_1(z)\hat{a}'(z))_{\geq 0} \\
& + \hat{\omega}_1(z)(\omega_2(z)a'(z))_{\geq 0} - \hat{\omega}_1(z)(\hat{\omega}_2(z)\hat{a}'(z))_{<0}]_{\leq M-1}
\end{aligned} \quad (2.59)$$

for any cotangent vectors  $\omega_i = (\omega_i(z), \hat{\omega}_i(z)) \in T_{\mathbf{a}}^* \mathcal{M}_{M,N}$  with  $i = 1, 2$ . Moreover, this multiplication has unity

$$\mathbf{e}^* := \left(0, \frac{z^M}{M\hat{v}_{-M}}\right) = \frac{1}{M}d\hat{h}^0. \quad (2.60)$$

*Proof.* The equality (2.59) is verified by using (2.10). Secondly, for any cotangent vector  $\omega = (\omega(z), \hat{\omega}(z)) \in T_{\mathbf{a}}^* \mathcal{M}_{M,N}$ , in consideration of the form of  $(a(z), \hat{a}(z))$  in (2.1) we have

$$\begin{aligned}
\mathbf{e}^* \cdot \omega &= z \left( \left[ -\omega(z) \left( \frac{z^M}{M\hat{v}_{-M}} \hat{a}'(z) \right)_{<0} \right]_{\geq -N}, \left[ -\hat{\omega}(z) \left( \frac{z^M}{M\hat{v}_{-M}} \hat{a}'(z) \right)_{<0} \right]_{\leq M-1} \right) \\
&= z \left( \left( \frac{\omega(z)}{z} \right)_{\geq -N}, \left( \frac{\hat{\omega}(z)}{z} \right)_{\leq M-1} \right) = \omega.
\end{aligned}$$

Thus we complete the proof of the lemma.  $\square$

As a combination of Lemmas 2.6 and 2.7, we obtain

**Proposition 2.8.** *The cotangent space  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$  carries a Frobenius algebra structure with the multiplication defined in (2.57), the unity  $\mathbf{e}^*$  given in (2.60) and the non-degenerate invariant bilinear form (2.11).*

Due to the bijection  $\eta$  in (2.4), we achieve the following result.

**Corollary 2.9.** *The tangent space  $T_{\mathbf{a}} \mathcal{M}_{m,n}$  is a Frobenius algebra with multiplication between any vectors  $\mathbf{X}_1$  and  $\mathbf{X}_2$  defined by*

$$\mathbf{X}_1 \cdot \mathbf{X}_2 := \eta(\eta^{-1}(\mathbf{X}_1) \cdot \eta^{-1}(\mathbf{X}_2)), \quad (2.61)$$

the unity vector

$$\mathbf{e} := \eta(\mathbf{e}^*) = (1, 1) = \frac{\partial}{\partial \hat{h}^M} \quad (2.62)$$

and the invariant inner product (2.28).

**2.3. The potential.** Now we proceed to compute the following symmetric 3-tensor

$$c(\partial_u, \partial_v, \partial_w) = \langle \partial_u \cdot \partial_v, \partial_w \rangle, \quad u, v, w \in \mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}} \quad (2.63)$$

where for simplicity we write  $\partial_u = \partial/\partial u \in T_{\mathbf{a}} \mathcal{M}_{M,N}$ . In the present section, unless otherwise stated the following convention will be assumed:

$$i, i_1, i_2, i_3 \in \mathbb{Z}, \quad j, j_1, j_2, j_3 \in \{1, 2, \dots, N-1\}, \quad k, k_1, k_2, k_3 \in \{0, 1, \dots, M\}.$$



**Lemma 2.10.** *The symmetric 3-tensor (2.63) is given by*

$$< \partial_{t^{i_1}} \cdot \partial_{h^{j_2}}, \partial_{\hat{h}^{k_3}} > = 0, \quad (2.64)$$

$$< \partial_{t^{i_1}} \cdot \partial_{t^{i_2}}, \partial_{h^{j_3}} > = \frac{-1}{2\pi \mathbf{i}} \oint_{|z|=1} z \left( \zeta'(z) \zeta(z)^{i_1+i_2} \right)_{<-1} \left( \chi'(z) \chi(z)^{N-1-j_3} \right)_{\geq 0} dz, \quad (2.65)$$

$$< \partial_{h^{j_1}} \cdot \partial_{h^{j_2}}, \partial_{t^{i_3}} > = \frac{-1}{2\pi N \mathbf{i}} \oint_{|z|=1} z \left( \zeta'(z) \zeta(z)^{i_3} \right)_{<0} \left( \chi'(z) \chi(z)^{N-1-j_1-j_2} \right)_{\geq -1} dz, \quad (2.66)$$

$$\begin{aligned} < \partial_{h^{j_1}} \cdot \partial_{h^{j_2}}, \partial_{\hat{h}^{k_3}} > \\ &= \frac{-1}{2\pi N \mathbf{i}} \oint_{|z|=1} z \left( \chi'(z) \chi(z)^{N-1-j_1-j_2} \right)_{\geq -1} \left( \chi'(z) \chi(z)^{M-1-k_3} \right)_{<0} dz, \end{aligned} \quad (2.67)$$

$$< \partial_{t^{i_1}} \cdot \partial_{t^{i_2}}, \partial_{\hat{h}^{k_3}} > = \frac{1}{2\pi \mathbf{i}} \oint_{|z|=1} z \left( \zeta'(z) \zeta(z)^{i_1+i_2} \right)_{\geq -1} \left( \hat{\chi}'(z) \hat{\chi}(z)^{M-1-k_3} \right)_{<0} dz, \quad (2.68)$$

$$< \partial_{\hat{h}^{k_1}} \cdot \partial_{\hat{h}^{k_2}}, \partial_{t^{i_3}} > = \frac{-1}{2\pi M \mathbf{i}} \oint_{|z|=1} z \left( \zeta'(z) \zeta(z)^{i_3} \right)_{\geq 0} \left( \hat{\chi}'(z) \hat{\chi}(z)^{M-1-k_1-k_2} \right)_{<-1} dz, \quad (2.69)$$

$$\begin{aligned} < \partial_{\hat{h}^{k_1}} \cdot \partial_{\hat{h}^{k_2}}, \partial_{h^{j_3}} > \\ &= \frac{-1}{2\pi M \mathbf{i}} \oint_{|z|=1} z \left( \chi'(z) \chi(z)^{N-1-j_3} \right)_{\geq 0} \left( \hat{\chi}'(z) \hat{\chi}(z)^{M-1-k_1-k_2} \right)_{<-1} dz, \end{aligned} \quad (2.70)$$

$$\begin{aligned} < \partial_{h^{j_1}} \cdot \partial_{h^{j_2}}, \partial_{h^{j_3}} > &= \frac{-1}{2\pi N^2 \mathbf{i}} \oint_{|z|=1} z \zeta'(z)_{<0} \left( \chi'(z) \chi(z)^{N-1-j_1-j_2-j_3} \right)_{\geq -1} dz \\ &\quad - \operatorname{res}_{z=\infty} \frac{z \left( \chi'(z) \chi(z)^{N-1-j_1} \right)_{\geq 0} \left( \chi'(z) \chi(z)^{N-1-j_2} \right)_{\geq 0} \left( \chi'(z) \chi(z)^{N-1-j_3} \right)_{\geq 0}}{l'(z)} dz, \end{aligned} \quad (2.71)$$

$$\begin{aligned} < \partial_{\hat{h}^{k_1}} \cdot \partial_{\hat{h}^{k_2}}, \partial_{\hat{h}^{k_3}} > &= \frac{1}{2\pi M^2 \mathbf{i}} \oint_{|z|=1} z \zeta'(z)_{\geq 0} \left( \hat{\chi}'(z) \hat{\chi}(z)^{M-1-k_1-k_2-k_3} \right)_{<-1} dz \\ &\quad + \operatorname{res}_{z=0} \frac{z \left( \hat{\chi}'(z) \chi(z)^{M-1-k_1} \right)_{<0} \left( \hat{\chi}'(z) \hat{\chi}(z)^{M-1-k_2} \right)_{<0} \left( \chi'(z) \hat{\chi}(z)^{M-1-k_3} \right)_{<0}}{l'(z)} dz, \end{aligned} \quad (2.72)$$

$$\begin{aligned} < \partial_{t^{i_1}} \cdot \partial_{t^{i_2}}, \partial_{t^{i_3}} > &= \frac{1}{2\pi \mathbf{i}} \oint_{|z|=1} z \zeta'(z) \zeta(z)^{i_1+i_2+i_3} \left( \frac{1}{2} \Pi[\zeta'(z)] - l'(z) \right) dz \\ &\quad - \frac{1}{4\pi \mathbf{i}} \oint_{|z|=1} z \zeta'(z) \left( \zeta(z)^{i_1+i_2} \Pi[\zeta'(z) \zeta(z)^{i_3}] \right. \\ &\quad \left. + \zeta(z)^{i_2+i_3} \Pi[\zeta'(z) \zeta(z)^{i_1}] + \zeta(z)^{i_3+i_1} \Pi[\zeta'(z) \zeta(z)^{i_2}] \right) dz, \end{aligned} \quad (2.73)$$

with the mapping  $\Pi[f(z)] = f(z)_{\geq 0} - f(z)_{< 0}$  in the final equality.

*Proof.* We denote

$$\eta_{uv} = \langle \partial_u, \partial_v \rangle,$$

of which the values are given in Proposition 2.4, and then we have

$$\langle \partial_u \cdot \partial_v, \partial_w \rangle = \eta_{u\tilde{u}} \eta_{v\tilde{v}} \eta_{w\tilde{w}} \langle d\tilde{u} \cdot d\tilde{v}, d\tilde{w} \rangle^* = \eta_{u\tilde{u}} \eta_{v\tilde{v}} \eta_{w\tilde{w}} \langle d\tilde{u} \cdot d\tilde{v}, \eta(d\tilde{w}) \rangle \quad (2.74)$$

with  $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in \mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$ . Let us compute the data in the right hand side.

First of all, one can check the following equalities

$$dt^i = \left( -(\zeta(z)^{-i-1})_{\geq -N+1}, (\zeta(z)^{-i-1})_{\leq M} \right), \quad (2.75)$$

$$dh^j = \left( (\chi(z)^{j-N})_{\geq -N+1}, 0 \right), \quad (2.76)$$

$$d\hat{h}^k = \left( 0, (\hat{\chi}(z)^{k-M})_{\leq M} \right). \quad (2.77)$$

Indeed, it follows from (1.17) that

$$\frac{\partial t^i}{\partial v_r} = -\frac{1}{2\pi i} \oint_{|p|=1} \zeta(p)^{-i-1} p^{r-1} dp, \quad \frac{\partial t^i}{\partial \hat{v}_s} = \frac{1}{2\pi i} \oint_{|p|=1} \zeta(p)^{-i-1} p^{s-1} dp,$$

hence

$$\begin{aligned} dt^i &= \sum_{r \leq N-1} \frac{\partial t^i}{\partial v_r} dv_r + \sum_{s \geq -M} \frac{\partial t^i}{\partial \hat{v}_s} d\hat{v}_s \\ &= \left( \sum_{r \leq N-1} \frac{\partial t^i}{\partial v_r} z^{-r}, \sum_{s \geq -M} \frac{\partial t^i}{\partial \hat{v}_s} z^{-s} \right) \\ &= \left( -(\zeta(z)^{-i-1})_{\geq -N+1}, (\zeta(z)^{-i-1})_{\leq M} \right). \end{aligned}$$

Similarly, the equalities (2.76)–(2.77) can be checked with the help of

$$\begin{aligned} \frac{\partial h^j}{\partial v_r} &= -\text{res}_{p=\infty} \chi(p)^{j-N} p^{r-1} dp, \quad \frac{\partial h^j}{\partial \hat{v}_s} = 0, \\ \frac{\partial \hat{h}^k}{\partial \hat{v}_s} &= \text{res}_{p=0} \hat{\chi}(p)^{k-M} p^{s-1} dp, \quad \frac{\partial \hat{h}^k}{\partial v_r} = 0 \end{aligned}$$

implied by (1.18)–(1.19).

Secondly, by using (2.59) we have

$$\begin{aligned} dt^{i_1} \cdot dt^{i_2} &= z \left( \left[ a'(z) \zeta(z)^{-i_1-i_2-2} - \zeta(z)^{-i_1-1} (\zeta(z)^{-i_2-1} \zeta'(z))_{< 0} \right. \right. \\ &\quad \left. \left. - \zeta(z)^{-i_2-1} (\zeta(z)^{-i_1-1} \zeta'(z))_{< 0} \right]_{\geq -N}, - \left[ \hat{a}'(z) \zeta(z)^{-i_1-i_2-2} \right. \right. \\ &\quad \left. \left. + \zeta(z)^{-i_1-1} (\zeta(z)^{-i_2-1} \zeta'(z))_{\geq 0} + \zeta(z)^{-i_2-1} (\zeta(z)^{-i_1-1} \zeta'(z))_{\geq 0} \right]_{\leq M-1} \right) \end{aligned} \quad (2.78)$$

with

$$a'(z) = \zeta'(z)_{< 0} + l'(z), \quad \hat{a}'(z) = -\zeta'(z)_{\geq 0} + l'(z),$$

and

$$\begin{aligned} dh^{j_1} \cdot dh^{j_2} = & \left( Nz \left[ \chi'(z) \chi(z)^{j_1+j_2-N-1} - \chi(z)^{j_1-N} \left( \chi'(z) \chi(z)^{j_2-1} \right)_{<0} \right. \right. \\ & \left. \left. - \chi(z)^{j_2-N} \left( \chi'(z) \chi(z)^{j_1-1} \right)_{<0} \right]_{\geq -N}, 0 \right), \end{aligned} \quad (2.79)$$

$$\begin{aligned} d\hat{h}^{k_1} \cdot d\hat{h}^{k_2} = & \left( 0, Mz \left[ -\hat{\chi}'(z) \hat{\chi}(z)^{k_1+k_2-M-1} + \hat{\chi}(z)^{k_1-M} \left( \hat{\chi}'(z) \hat{\chi}(z)^{k_2-1} \right)_{\geq 0} \right. \right. \\ & \left. \left. + \hat{\chi}(z)^{k_2-M} \left( \hat{\chi}'(z) \hat{\chi}(z)^{k_1-1} \right)_{\geq 0} \right]_{\leq M-1} \right). \end{aligned} \quad (2.80)$$

Thirdly, substituting (2.75)–(2.77) into (2.14) we obtain

$$\eta(dt^i) = z \left( (\zeta'(z) \zeta(z)^{-i-1})_{<0}, -(\zeta'(z) \zeta(z)^{-i-1})_{\geq 0} \right), \quad (2.81)$$

$$\eta(dh^j) = Nz \left( (\chi'(z) \chi(z)^{j-1})_{\geq 0}, (\chi'(z) \chi(z)^{j-1})_{\geq 0} \right), \quad (2.82)$$

$$\eta(d\hat{h}^k) = -Mz \left( (\hat{\chi}'(z) \hat{\chi}(z)^{k-1})_{<0}, (\hat{\chi}'(z) \hat{\chi}(z)^{k-1})_{<0} \right). \quad (2.83)$$

Finally, we substitute the above data into (2.74), and conclude the lemma after a tedious by straightforward computation.  $\square$

**Lemma 2.11** ([17]). *There exists a function  $F_{M,N}$  depending polynomially on  $\mathbf{h} \cup \hat{\mathbf{h}} \cup \{e^{\hat{h}^0}\}$  such that*

$$\frac{\partial^3 F_{M,N}}{\partial u \partial v \partial w} = -(\text{res}_{z=\infty} + \text{res}_{z=0}) \frac{\partial_u l(z) \cdot \partial_v l(z) \cdot \partial_w l(z)}{z^2 l'(z)} dz \quad (2.84)$$

for any  $u, v, w \in \mathbf{h} \cup \hat{\mathbf{h}}$ .

We introduce a function of  $\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$  as

$$\begin{aligned} \mathcal{F}_{M,N} = & \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( \frac{1}{2} \zeta(z_1) \zeta(z_2) - \zeta(z_1) l(z_2) + l(z_1) \zeta(z_2) \right) \times \\ & \times \log \left( \frac{z_2 - z_1}{z_2} \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} + \left( \frac{1}{2} t^{-1} - \hat{h}^M \right) \sum_{i \geq 0} t^i t^{-i-1} + F_{M,N}, \end{aligned} \quad (2.85)$$

where

$$\log \left( \frac{z_2 - z_1}{z_2} \right) = - \sum_{m \geq 1} \frac{1}{m} \left( \frac{z_1}{z_2} \right)^m.$$

**Proposition 2.12.** *The function  $\mathcal{F}_{M,N}$  satisfies*

$$c(\partial_u, \partial_v, \partial_w) = \frac{\partial^3 \mathcal{F}_{M,N}}{\partial u \partial v \partial w}, \quad u, v, w \in \mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}. \quad (2.86)$$

Consequently, letting  $\nabla$  denote the Levi-Civita connection of the metric  $<, >$ , the 4-tensor  $\nabla_{\partial_s} c(\partial_u, \partial_v, \partial_w)$  is symmetric with respect to the flat coordinates  $s, u, v$  and  $w$ .

*Proof.* In this proof we use the notation  $f^*(z) = zf'(z)$  for a differentiable function  $f(z)$ . If  $f(z)$  and  $g(z)$  are holomorphic (single-valued unless otherwise stated) on a neighborhood of  $|z| = 1$ , then one has

$$f^*(z)_{\geq 0} = f^*(z)_{>0} = zf'(z)_{>0}, \quad f^*(z)_{\leq 0} = f^*(z)_{<0} = zf'(z)_{<0}$$

and the following formula of integration by parts:

$$\frac{1}{2\pi i} \oint_{|z|=1} f^*(z)g(z) \frac{dz}{z} = -\frac{1}{2\pi i} \oint_{|z|=1} f(z)g^*(z) \frac{dz}{z}. \quad (2.87)$$

Let

$$Y_i(z) = \begin{cases} \frac{\zeta(z)^{i+1}}{i+1}, & i \in \mathbb{Z} \setminus \{-1\}; \\ \log \zeta(z), & i = -1. \end{cases} \quad (2.88)$$

Clearly all  $Y_i(z)$  and  $Y_i^*(z)$  are holomorphic at  $|z| = 1$  except  $Y_{-1}(z)$  being multi-valued, hence one has

$$\frac{1}{2\pi i} \oint_{|z|=1} Y_i^*(z) \frac{dz}{z} = \frac{\delta_{i,-1}}{2\pi i} \oint_{|z|=1} \frac{\zeta'(z)}{\zeta(z)} dz = \delta_{i,-1}.$$

By using Lemma 2.5, it is easy to see

$$\frac{\partial \zeta(z)}{\partial t^i} = -z\zeta'(z)\zeta(z)^i = -Y_i^*(z), \quad (2.89)$$

$$\frac{\partial^2 \zeta(z)}{\partial t^{i_1} \partial t^{i_2}} = Y_{i_1+i_2}^{**}(z), \quad \frac{\partial^3 \zeta(z)}{\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}} = -Y_{i_1+i_2+i_3}^{***}(z). \quad (2.90)$$

We want to compute the third order derivatives of  $\mathcal{F}$  with respect to  $\mathbf{t} \cup \mathbf{h} \cup \hat{\mathbf{h}}$ . To this end, let  $\tilde{\mathcal{F}}$  denote the integral part on the right hand side of (2.85), then we have

$$\begin{aligned} & \frac{\partial^3 \tilde{\mathcal{F}}}{\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}} \\ &= \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( -\frac{1}{2} Y_{i_1+i_2+i_3}^{***}(z_1) \zeta(z_2) - \frac{1}{2} \zeta(z_1) Y_{i_1+i_2+i_3}^{***}(z_2) \right. \\ & \quad \left. - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} (Y_{i_1+i_2}^{**}(z_1) Y_{i_3}^*(z_2) + Y_{i_1}^*(z_1) Y_{i_2+i_3}^{**}(z_2)) \right. \\ & \quad \left. + Y_{i_1+i_2+i_3}^{***}(z_1) l(z_2) - l(z_1) Y_{i_1+i_2+i_3}^{***}(z_2) \right) \log \left( \frac{z_2 - z_1}{z_2} \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\ &= \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( -\frac{1}{2} Y_{i_1+i_2+i_3}^{**}(z_1) \zeta(z_2) + \frac{1}{2} \zeta(z_1) Y_{i_1+i_2+i_3}^{**}(z_2) \right. \\ & \quad \left. - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} (Y_{i_1+i_2}^*(z_1) Y_{i_3}^*(z_2) - Y_{i_1}^*(z_1) Y_{i_2+i_3}^*(z_2)) \right. \\ & \quad \left. + Y_{i_1+i_2+i_3}^{**}(z_1) l(z_2) + l(z_1) Y_{i_1+i_2+i_3}^{**}(z_2) \right) \frac{z_1}{z_2 - z_1} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \oint \left( -\frac{1}{2} Y_{i_1+i_2+i_3}^{**}(z_2)_{<0} \zeta(z_2) + \frac{1}{2} \zeta(z_2)_{<0} Y_{i_1+i_2+i_3}^{**}(z_2) \right. \\
&\quad \left. - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} (Y_{i_1+i_2}^*(z_2)_{<0} Y_{i_3}^*(z_2) - Y_{i_1}^*(z_2)_{<0} Y_{i_2+i_3}^*(z_2)) \right. \\
&\quad \left. + Y_{i_1+i_2+i_3}^{**}(z_2)_{<0} l(z_2) + l(z_2)_{<0} Y_{i_1+i_2+i_3}^{**}(z_2) \right) \frac{dz_2}{z_2} \\
&= \frac{1}{2\pi i} \oint \left( Y_{i_1+i_2+i_3}^*(z) \left( \frac{1}{2} \zeta^*(z)_{>0} - \frac{1}{2} \zeta^*(z)_{<0} - l^*(z)_{>0} - l^*(z)_{<0} \right) \right. \\
&\quad \left. - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} Y_{i_1+i_2}^*(z) (Y_{i_3}^*(z)_{>0} - Y_{i_3}^*(z)_{<0}) \right) \frac{dz}{z} \\
&= \frac{1}{2\pi i} \oint \left( Y_{i_1+i_2+i_3}^*(z) \left( \frac{1}{2} \Pi \zeta^*(z) - l^*(z) \right) \right. \\
&\quad \left. - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} Y_{i_1+i_2}^*(z) z \Pi Y_{i_3}'(z) \right) \frac{dz}{z} \\
&\quad - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} \frac{1}{2\pi i} \oint Y_{i_1+i_2}^*(z) \frac{dz}{z} \cdot \frac{1}{2\pi i} \oint Y_{i_3}^*(z) \frac{dz}{z} \\
&= \langle \partial_{t^{i_1}} \cdot \partial_{t^{i_2}}, \partial_{t^{i_3}} \rangle - \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} \delta_{i_1+i_2, -1} \delta_{i_3, -1}, \tag{2.91}
\end{aligned}$$

where “c.p.” means “cyclic permutation”, in the second and the fourth equalities the formula of integration by parts is used. Hence in consideration of  $\partial_{t^i} F_{M,N} = 0$  we obtain

$$\begin{aligned}
\frac{\partial^3 \mathcal{F}_{M,N}}{\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}} &= \frac{\partial^3 \tilde{\mathcal{F}}}{\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}} + \frac{1}{2} \sum_{\text{c.p.}(i_1, i_2, i_3)} \delta_{i_1+i_2, -1} \delta_{i_3, -1} \\
&= c(\partial_{t^{i_1}}, \partial_{t^{i_2}}, \partial_{t^{i_3}}). \tag{2.92}
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\frac{\partial^3 \tilde{\mathcal{F}}}{\partial t^{i_1} \partial t^{i_2} \partial \hat{h}^{k_3}} \\
&= \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( Y_{i_1+i_2}^{**}(z_1) (z_2 \hat{\chi}'(z_2) \hat{\chi}(z_2)^{M-k_3-1})_{\leq 0} \right. \\
&\quad \left. - (z_1 \hat{\chi}'(z_1) \hat{\chi}(z_1)^{M-k_3-1})_{\leq 0} Y_{i_1+i_2}^{**}(z_2) \right) \log \left( \frac{z_2 - z_1}{z_2} \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \\
&= \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( Y_{i_1+i_2}^*(z_1) (z_2 \hat{\chi}'(z_2) \hat{\chi}(z_2)^{M-k_3-1})_{\leq 0} \right. \\
&\quad \left. + (z_1 \hat{\chi}'(z_1) \hat{\chi}(z_1)^{M-k_3-1})_{\leq 0} Y_{i_1+i_2}^*(z_2) \right) \frac{z_1}{z_2 - z_1} \frac{dz_1}{z_1} \frac{dz_2}{z_2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \oint \left( Y_{i_1+i_2}^*(z_2)_{<0} (z_2 \hat{\chi}'(z_2) \hat{\chi}(z_2)^{M-k_3-1})_{\leq 0} \right. \\
&\quad \left. + (z_2 \hat{\chi}'(z_2) \hat{\chi}(z_2)^{M-k_3-1})_{<0} Y_{i_1+i_2}^*(z_2) \right) \frac{dz_2}{z_2} \\
&= \frac{1}{2\pi i} \oint Y_{i_1+i_2}^*(z)_{>0} (z \hat{\chi}'(z) \hat{\chi}(z)^{M-k_3-1})_{<0} \frac{dz}{z} \\
&= \frac{1}{2\pi i} \oint Y_{i_1+i_2}^*(z)_{\geq 0} (z \hat{\chi}'(z) \hat{\chi}(z)^{M-k_3-1})_{\leq 0} \frac{dz}{z} \\
&\quad - \frac{1}{2\pi i} \oint Y_{i_1+i_2}^*(z) \frac{dz}{z} \cdot \frac{1}{2\pi i} \oint z \hat{\chi}'(z) \hat{\chi}(z)^{M-k_3-1} \frac{dz}{z} \\
&= \langle \partial_{t^{i_1}} \cdot \partial_{t^{i_2}}, \partial_{\hat{h}^{k_3}} \rangle + \delta_{i_1+i_2, -1} \delta_{k_3, M}, \tag{2.93}
\end{aligned}$$

which leads to

$$\frac{\partial^3 \mathcal{F}_{M,N}}{\partial t^{i_1} \partial t^{i_2} \partial \hat{h}^{k_3}} = c(\partial_{t^{i_1}}, \partial_{t^{i_2}}, \partial_{\hat{h}^{k_3}}).$$

The other cases are similar. Therefore the proposition is proved.  $\square$

**Proposition 2.13.** *The function  $\mathcal{F}_{M,N}$  can be written in a coordinate free way as*

$$\begin{aligned}
\mathcal{F}_{M,N} &= \frac{1}{(2\pi i)^2} \oint \oint_{|z_1| < |z_2|} \left( \frac{1}{2} \zeta(z_1) \zeta(z_2) - \zeta(z_1) l(z_2) + l(z_1) \zeta(z_2) \right) \times \\
&\quad \times \log \left( \frac{z_2 - z_1}{z_2} \right) \frac{dz_1}{z_1} \frac{dz_2}{z_2} - \frac{1}{(2\pi i)^2} \oint_{|z|=1} \left( \frac{1}{2} \zeta(z) + l(z) \right) \frac{dz}{z} \times \\
&\quad \times \oint_{|z|=1} \zeta(z) \left( \log \frac{\zeta(z)}{z} - 1 \right) \frac{dz}{z} + F_{M,N}. \tag{2.94}
\end{aligned}$$

*Proof.* According to (1.17)–(1.18) one has

$$\frac{1}{2\pi i} \oint \left( \frac{1}{2} \zeta(z) + l(z) \right) \frac{dz}{z} = -\frac{1}{2} t^{-1} + \hat{h}^M.$$

Comparing (2.94) with (2.85), we only need to show

$$\frac{1}{2\pi i} \oint \zeta(z) \left( \log \frac{\zeta(z)}{z} - 1 \right) \frac{dz}{z} = \sum_{i \geq 0} t^i t^{-i-1}. \tag{2.95}$$

Indeed, starting from the right hand side,

$$\begin{aligned}
\text{r.h.s.} &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{2} \left( \log \frac{z(\zeta)}{\zeta} \right)^2 d\zeta \\
&= -\frac{1}{2\pi i} \oint_{|z|=1} \zeta(z) \log \frac{z}{\zeta(z)} \left( \frac{1}{z} - \frac{\zeta'(z)}{\zeta(z)} \right) dz \\
&= \frac{1}{2\pi i} \oint_{|z|=1} \zeta(z) \log \frac{\zeta(z)}{z} \frac{dz}{z} + \frac{1}{2\pi i} \oint_{\Gamma} \log \frac{z(\zeta)}{\zeta} d\zeta \\
&= \frac{1}{2\pi i} \oint_{|z|=1} \zeta(z) \log \frac{\zeta(z)}{z} \frac{dz}{z} + t^{-1} = \text{l.h.s.} \tag{2.96}
\end{aligned}$$

Thus the proposition is proved.  $\square$

**2.4. The Euler vector field.** We are to fix a Euler vector field on  $\mathcal{M}_{M,N}$  and show the quasi-homogeneity property of  $\mathcal{F}_{M,N}$ , which will imply that  $\mathcal{M}_{M,N}$  is really a Frobenius manifold.

Let us assign a degree to each of the flat coordinates:

$$\deg t^i = -i, \quad i \in \mathbb{Z} \setminus \{0\}; \quad \deg e^{t^0} = \frac{1}{N} - 1; \quad (2.97)$$

$$\deg h^j = \frac{j}{N}, \quad 1 \leq j \leq N-1; \quad (2.98)$$

$$\deg e^{\hat{h}^0} = 1 + \frac{M}{N}; \quad \deg \hat{h}^k = \frac{k}{M}, \quad 1 \leq k \leq M \quad (2.99)$$

and introduce the following vector field

$$\begin{aligned} \mathcal{E}_{M,N} = & - \sum_{i \in \mathbb{Z} \setminus \{0\}} i t^i \frac{\partial}{\partial t^i} - \frac{N-1}{N} \frac{\partial}{\partial t^0} + \sum_{j=1}^{N-1} \frac{j}{N} h^j \frac{\partial}{\partial h^j} + \\ & + \sum_{k=1}^M \frac{k}{M} \hat{h}^k \frac{\partial}{\partial \hat{h}^k} + \left(1 + \frac{M}{N}\right) \frac{\partial}{\partial \hat{h}^0}. \end{aligned} \quad (2.100)$$

**Proposition 2.14.** *The function  $\mathcal{F}_{M,N}$  satisfies*

$$\text{Lie}_{\mathcal{E}_{M,N}} \mathcal{F}_{M,N} = 2\mathcal{F}_{M,N} + \text{quadratic terms in flat coordinates.} \quad (2.101)$$

*Proof.* If we assume  $\deg z = 1/N$  besides (2.97)–(2.99), then each of the functions  $\zeta(z)$  and  $l(z)$  are homogeneous of degree 1, hence  $\mathcal{F}_{M,N}$  given in (2.94) has degree 2 modulo quadratic terms. The proposition is proved.  $\square$

Up to now, we have shown that  $\mathcal{M}_{M,N}$  is a Frobenius manifold, on which the potential is  $\mathcal{F}_{M,N}$ , the unity vector field is  $\mathbf{e} = \partial/\partial \hat{h}^M$  and the Euler vector field is  $\mathcal{E}_{M,N}$ .

From the proof of the above proposition, one also sees

$$\left( \mathcal{E}_{M,N} + \frac{z}{N} \frac{\partial}{\partial z} \right) \vartheta(z) = \vartheta(z) \quad (2.102)$$

for  $\vartheta(z) \in \{a(z), \hat{a}(z), \zeta(z), l(z)\}$ . Hence  $\mathcal{E}_{M,N}$  is equal to the following vector field

$$\mathcal{E}_{M,N} = \sum_{r \leq N-1} \left(1 - \frac{r}{N}\right) v_r \frac{\partial}{\partial v_r} + \sum_{s \geq -M} \left(1 - \frac{s}{N}\right) \hat{v}_s \frac{\partial}{\partial \hat{v}_s}. \quad (2.103)$$

By using (2.3), one can represent  $\mathcal{E}_{M,N}$  to the form of Laurent series as

$$\mathcal{E}_{M,N} = \left( a(z) - \frac{z}{N} a'(z), \hat{a}(z) - \frac{z}{N} \hat{a}'(z) \right). \quad (2.104)$$

We remark that one can start from the formula (2.104), then deduce (2.100) and (2.101) with the help of relevant result in [17].

**2.5. Proof of Main Theorem.** According to Corollary 2.9, Propositions 2.4 and 2.12–2.14, one sees that  $\mathcal{M}_{M,N}$  is a Frobenius manifold with properties required by the Main Theorem in Section 1. To prove the Main Theorem, we only need to show the semisimplicity of  $\mathcal{M}_{M,N}$ .

With the same method as in [7], let

$$du(p) = \frac{\hat{a}'(p)}{\zeta'(p)} da(p) - \frac{a'(p)}{\zeta'(p)} d\hat{a}(p), \quad p \in S^1 \quad (2.105)$$

which is a generating function for a basis of the cotangent space  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$ . By using (2.11) and (2.57), one can check

$$\langle du(p), du(q) \rangle^* = f(p) \delta(p - q), \quad (2.106)$$

$$du(p) \cdot du(q) = f(p) \delta(p - q) du(p), \quad (2.107)$$

where

$$f(p) = -p^2 \frac{a'(p) \hat{a}'(p)}{\zeta'(p)}$$

and  $\delta(p - q) = \sum_{k \in \mathbb{Z}} (p^k / q^{k+1})$  is the formal delta function such that

$$\frac{1}{2\pi i} \oint_{|q|=1} f(q) \delta(p - q) dq = f(p).$$

The formula (2.107) implies that at a generic points there is no nilpotent elements in the Frobenius algebra  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$ , hence the Frobenius algebra  $T_{\mathbf{a}}^* \mathcal{M}_{M,N}$  is semisimple and so is  $T_{\mathbf{a}} \mathcal{M}_{M,N}$ . That is to say, the Frobenius manifold  $\mathcal{M}_{M,N}$  is semisimple.

Therefore we complete the proof of the Main Theorem.  $\square$

**2.6. The intersection form.** At the end of this section, let us consider the intersection form of  $\mathcal{M}_{M,N}$ , which is closely related to the theory of integrable hierarchies. Being analogous to finite-dimensional cases, we define the intersection form on the cotangent space  $T_{\mathbf{a}}^* \mathcal{M}_{m,n}$  by

$$(d\alpha(p), d\beta(q))^* := i_{\mathcal{E}_{M,N}}(d\alpha(p) \cdot d\beta(q)), \quad \alpha, \beta \in \{a, \hat{a}\}. \quad (2.108)$$

**Proposition 2.15.** *For  $\alpha, \beta \in \{a, \hat{a}\}$  it holds that*

$$(d\alpha(p), d\beta(q))^* = \frac{pq}{p - q} \left( \alpha'(p) B(\beta(q)) - B(\alpha(p)) \beta'(q) \right), \quad (2.109)$$

where  $B(\alpha(p)) = \alpha(p) - \frac{p}{N} \alpha'(p)$ .

*Proof.* Substituting (2.57) into (2.108) and using (2.102), we have

$$\begin{aligned} (d\alpha(p), d\beta(q))^* &= \langle \mathcal{E}_{M,N}, d\alpha(p) \cdot d\beta(q) \rangle \\ &= \frac{pq}{p - q} (\alpha'(p) \langle \mathcal{E}_{M,N}, d\beta(q) \rangle - \beta'(q) \langle \mathcal{E}_{M,N}, d\alpha(p) \rangle) \\ &= \frac{pq}{p - q} \left( \alpha'(p) \left[ \beta(q) - \frac{q}{N} \beta'(q) \right] - \beta'(q) \left[ \alpha(p) - \frac{p}{N} \alpha'(p) \right] \right), \end{aligned}$$



$$= \frac{pq}{p-q} [\alpha'(p)\beta(q) - \alpha(p)\beta'(q)] + \frac{pq}{N} \alpha'(p)\beta'(q)$$

which is exactly (2.109). The proposition is proved.  $\square$

The intersection form, in the same way as for the flat metric (2.11), induces a linear map

$$g : T_{\mathbf{a}}^* \mathcal{M}_{M,N} \rightarrow T_{\mathbf{a}} \mathcal{M}_{M,N} \quad (2.110)$$

such that

$$\langle \omega_1, g(\omega_2) \rangle = (\omega_1, \omega_2)^* \quad (2.111)$$

for any  $\omega_1, \omega_2 \in T_{\mathbf{a}}^* \mathcal{M}_{M,N}$ .

**Lemma 2.16.** *The linear map  $g$  defined by (2.110)–(2.111) is a bijection.*

*Proof.* The proof is very similar with that of Lemma 2.2.

On the one hand, according to (2.9) one has

$$g(d\beta(q)) = \left( (d\beta(q), da(z))^*, (d\beta(q), d\hat{a}(z))^* \right), \quad \beta \in \{a, \hat{a}\}.$$

Hence for arbitrary  $\omega = (\omega(z), \hat{\omega}(z)) \in T_{\mathbf{a}}^* \mathcal{M}_{m,n}$ , by using (2.10) we have

$$\begin{aligned} g(\omega)(z) &= \frac{1}{2\pi i} \oint_{|q|=1} [\omega(q)g(da(q)) + \hat{\omega}(q)g(d\hat{a}(q))] \frac{dq}{q} \\ &= \left( za'(z)[\omega(z)B(a(z)) + \hat{\omega}(z)B(\hat{a}(z))]_{<0} \right. \\ &\quad \left. - zB(a(z))[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{<0}, \right. \\ &\quad \left. - z\hat{a}'(z)[\omega(z)B(a(z)) + \hat{\omega}(z)B(\hat{a}(z))]_{\geq 0} \right. \\ &\quad \left. + zB(\hat{a}(z))[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{\geq 0} \right) \end{aligned} \quad (2.112)$$

which implies that  $g$  is surjective.

On the other hand, given any  $\mathbf{X} = (X(z), \hat{X}(z)) \in T_{\mathbf{a}} \mathcal{M}_{M,N}$ , we want to solve the equation

$$\mathbf{X} = g(\omega),$$

i.e.,

$$\begin{aligned} X(z) &= za'(z)[\omega(z)B(a(z)) + \hat{\omega}(z)B(\hat{a}(z))]_{<0} \\ &\quad - zB(a(z))[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{<0}, \\ \hat{X}(z) &= -z\hat{a}'(z)[\omega(z)B(a(z)) + \hat{\omega}(z)B(\hat{a}(z))]_{\geq 0} \\ &\quad + zB(\hat{a}(z))[\omega(z)a'(z) + \hat{\omega}(z)\hat{a}'(z)]_{\geq 0}. \end{aligned}$$

Denote

$$\Theta(z) = \frac{\hat{a}'(z)X(z) - a'(z)\hat{X}(z)}{z[a(z)\hat{a}'(z) - a'(z)\hat{a}(z)]}. \quad (2.113)$$

It is straightforward to check

$$\Theta(z) = (\omega(z)a'(z))_{\geq 0} - (\hat{\omega}(z)\hat{a}'(z))_{<0}. \quad (2.114)$$

Observe

$$\frac{1}{a'(z)} = \left( \frac{1}{a'(z)} \right)_{\leq -N+1}, \quad \frac{1}{\hat{a}'(z)} = \left( \frac{1}{\hat{a}'(z)} \right)_{\geq M+1}$$

then by using (2.114) we obtain

$$\omega(z) = \omega(z)_{>-N} = \left( \frac{1}{a'(z)} (\omega(z)a'(z))_{\geq 0} \right)_{>-N} = \left( \frac{1}{a'(z)} \Theta(z)_{\geq 0} \right)_{>-N}, \quad (2.115)$$

$$\hat{\omega}(z) = \hat{\omega}(z)_{\leq M} = \left( \frac{1}{\hat{a}'(z)} (\hat{\omega}(z)\hat{a}'(z))_{<0} \right)_{\leq M} = - \left( \frac{1}{\hat{a}'(z)} \Theta(z)_{<0} \right)_{\leq M}. \quad (2.116)$$

It follows that  $g$  is injective. The lemma is proved.  $\square$

With the help of the bijection  $g$ , a bilinear form on the tangent space  $T_{\mathbf{a}}\mathcal{M}_{M,N}$  can be defined as

$$(\partial_1, \partial_2) := \langle g^{-1}(\partial_1), \partial_2 \rangle = (g^{-1}(\partial_1), g^{-1}(\partial_2))^*. \quad (2.117)$$

By using (2.115)–(2.116), a short computation leads to

**Proposition 2.17.** *For any  $\partial_1, \partial_2 \in T_{\mathbf{a}}\mathcal{M}_{M,N}$ ,*

$$(\partial_1, \partial_2) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\left( \frac{\partial_1 a(z)}{a'(z)} - \frac{\partial_1 \hat{a}(z)}{\hat{a}'(z)} \right) \cdot \left( \frac{\partial_2 a(z)}{a'(z)} - \frac{\partial_2 \hat{a}(z)}{\hat{a}'(z)} \right) dz}{\frac{a(z)}{a'(z)} - \frac{\hat{a}(z)}{\hat{a}'(z)}} \frac{dz}{z^2}. \quad (2.118)$$

### 3. RELATION TO DISPERSIONLESS TODA LATTICE HIERARCHY

Let us study how the infinite-dimensional Frobenius manifolds constructed above are related to the Toda lattice hierarchy.

In this section we work on the loop space, say,  $\mathcal{LM}_{M,N}$ , of smooth maps from  $S^1$  to  $\mathcal{M}_{M,N}$ . More precisely, the space  $\mathcal{LM}_{M,N}$  consists of points  $\mathbf{a} = (a(z, x), \hat{a}(z, x))$  of the form (1.13) with coefficients  $v_i$  and  $\hat{v}_j$  being smooth functions of  $x \in S^1$  constrained by the conditions (M1)–(M3). The tangent and the cotangent spaces of  $\mathcal{LM}_{M,N}$  are identified with spaces of Laurent series in a natural way as (2.2) and (2.4), respectively. Thereby the pairing between a covector  $\boldsymbol{\omega} = (\omega(z, x), \hat{\omega}(z, x)) \in T_{\mathbf{a}}^*\mathcal{LM}_{M,N}$  and a vector  $\mathbf{X} = (X(z, x), \hat{X}(z, x)) \in T_{\mathbf{a}}\mathcal{LM}_{M,N}$  reads (cf. (2.5))

$$\langle \boldsymbol{\omega}, \mathbf{X} \rangle = \frac{1}{2\pi i} \oint_{x \in S^1} \oint_{|z|=1} [\omega(z, x)X(z, x) + \hat{\omega}(z, x)\hat{X}(z, x)] \frac{dz}{z} dx. \quad (3.1)$$

To avoid lengthy notations, we will write  $\alpha(z)$  instead of  $\alpha(z, x)$  whenever no confusion would happen.

Let

$$\lambda(z) = a(z)^{1/N} = z + \frac{1}{N}v_{N-1} + O(z^{-1}), \quad |z| \rightarrow \infty; \quad (3.2)$$

$$\hat{\lambda}(z) = \hat{a}(z)^{1/M} = \hat{v}_{-M}^{1/M} z^{-1} + \frac{\hat{v}_{-M+1}}{M\hat{v}_{-M}^{(M-1)/M}} + O(z), \quad |z| \rightarrow 0. \quad (3.3)$$

A hierarchy of evolutionary equations on the loop space  $\mathcal{LM}_{m,n}$  is defined as follows:

$$\frac{\partial a(z)}{\partial s_n} = \{(\lambda(z)^n)_{\geq 0}, a(z)\}, \quad \frac{\partial \hat{a}(z)}{\partial s_n} = \{(\lambda(z)^n)_{\geq 0}, \hat{a}(z)\}, \quad (3.4)$$

$$\frac{\partial a(z)}{\partial \hat{s}_n} = \{-(\hat{\lambda}(z)^n)_{< 0}, a(z)\}, \quad \frac{\partial \hat{a}(z)}{\partial \hat{s}_n} = \{-(\hat{\lambda}(z)^n)_{< 0}, \hat{a}(z)\}, \quad (3.5)$$

where  $n = 1, 2, 3, \dots$  and the Lie bracket reads

$$\{f, g\} := z \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial x} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial x} \right).$$

This hierarchy is the dispersionless limit of the Toda lattice hierarchy [22, 21].

In fact, equations (3.4)–(3.5) can be defined equivalently with  $(a(z), \hat{a}(z))$  replaced by  $(\lambda(z), \hat{\lambda}(z))$ . Based on this fact, denote  $\hat{v}_{-M}^{1/M} = e^u$ , then equations (3.4)–(3.5) yields

$$\partial_{s_1} e^u = \frac{1}{N} e^u \partial_x v_{N-1}, \quad \frac{1}{N} \partial_{\hat{s}_1} v_{N-1} = \partial_x e^u.$$

Eliminating  $v_{N-1}$ , one has

$$\partial_{s_1} \partial_{\hat{s}_1} u = \partial_x^2 e^u, \quad (3.6)$$

which is rewritten to (1.1) with  $t = s_1 + \hat{s}_1$  and  $y = s_1 - \hat{s}_1$ .

Consider the following space of local functionals on  $\mathcal{LM}_{M,N}$ :

$$\mathcal{F} = \left\{ F = \int_{S^1} f(v_{N-1}(x), v_{N-2}(x), \dots, \hat{v}_{-M}(x), \hat{v}_{-M+1}(x), \dots) dx \right\}.$$

For any  $F \in \mathcal{F}$ , its variational gradient  $dF$  is a covector field on  $\mathcal{LM}_{M,N}$  such that  $\delta F = \langle dF, \delta \mathbf{a} \rangle$ .

As a result of [24], the loop space  $\mathcal{LM}_{M,N}$  is equipped with two compatible Poisson brackets

$$\{F, H\}_\nu = \langle dF, \mathcal{P}_\nu(dH) \rangle, \quad \nu = 1, 2 \quad (3.7)$$

for any  $F, H \in \mathcal{F}$ , where

$$\begin{aligned} & \mathcal{P}_1(\omega(z), \hat{\omega}(z)) \\ &= \left( -\{(\omega(z) + \hat{\omega}(z))_{< 0}, a(z)\} + (\{\omega(z), a(z)\} + \{\hat{\omega}(z), \hat{a}(z)\})_{\leq 0}, \right. \\ & \quad \left. \{(\omega(z) + \hat{\omega}(z))_{\geq 0}, \hat{a}(z)\} - (\{\omega(z), a(z)\} + \{\hat{\omega}(z), \hat{a}(z)\})_{> 0} \right), \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \mathcal{P}_2(\omega(z), \hat{\omega}(z)) \\ &= \left( -\{(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z))_{< 0}, a(z)\} + a(z)(\{\omega(z), a(z)\} + \{\hat{\omega}(z), \hat{a}(z)\})_{\leq 0}, \right. \\ & \quad \left. \{(a(z)\omega(z) + \hat{a}(z)\hat{\omega}(z))_{\geq 0}, \hat{a}(z)\} - \hat{a}(z)(\{\omega(z), a(z)\} + \{\hat{\omega}(z), \hat{a}(z)\})_{> 0} \right) \\ & \quad + (za'(z)\partial_x f, z\hat{a}'(z)\partial_x f) \end{aligned} \quad (3.9)$$

with

$$f = \frac{1}{N} \frac{1}{2\pi i} \oint_{|z|=1} (a'(z)\omega(z) + \hat{a}'(z)\hat{\omega}(z)) dz.$$

Observe the slight difference between these Poisson structures and those given in [5] and [7] for  $M = N = 1$ .

**Proposition 3.1** ([24]). *The dispersionless Toda lattice hierarchy (3.4)–(3.5) can be represented to a bi-hamiltonian form as*

$$\frac{\partial F}{\partial s_n} = \{F, H_{n+N}\}_1 = \{F, H_n\}_2, \quad \frac{\partial F}{\partial \hat{s}_n} = \{F, \hat{H}_{n+M}\}_1 = \{F, \hat{H}_n\}_2 \quad (3.10)$$

with  $n = 1, 2, 3, \dots$  and

$$H_n = \frac{N}{n} \frac{1}{2\pi i} \oint_{S^1} \oint_{|z|=1} \lambda(z)^n \frac{dz}{z} dx, \quad \hat{H}_n = \frac{M}{n} \frac{1}{2\pi i} \oint_{S^1} \oint_{|z|=1} \hat{\lambda}(z)^n \frac{dz}{z} dx. \quad (3.11)$$

**Proposition 3.2.** *The bi-hamiltonian structure (3.8)–(3.9) coincides with the one induced by the pencil consisting of the metric (2.11) and the intersection form (2.109) on  $\mathcal{M}_{M,N}$ .*

*Proof.* We introduce two generating functions for local functionals

$$a(p, y) = p^N + \sum_{i \leq N-1} v_i(y) p^i, \quad \hat{a}(p, y) = \sum_{j \geq -M} \hat{v}_j(y) p^j. \quad (3.12)$$

Their variational gradients are (cf. (2.7)–(2.8)) respectively

$$da(p, y) = \left( \frac{p^N}{z^{N-1}(p-z)} \delta(x-y), 0 \right), \quad |z| < |p|; \quad (3.13)$$

$$d\hat{a}(p, y) = \left( 0, \frac{z^{M+1}}{p^M(z-p)} \delta(x-y) \right), \quad |z| > |p|. \quad (3.14)$$

Substituting them into (3.7), then by a straightforward calculation we have

$$\begin{aligned} \{\alpha(p, x), \beta(q, y)\}_1 &= \frac{pq}{p-q} (\alpha'(p) - \beta'(q)) \delta'(x-y) \\ &\quad + pq \left( \frac{\partial_x \alpha(p) - \partial_x \beta(q)}{(p-q)^2} - \frac{\partial_x \beta'(q)}{p-q} \right) \delta(x-y), \end{aligned} \quad (3.15)$$

$$\begin{aligned} \{\alpha(p, x), \beta(q, y)\}_2 &= pq \left( \frac{\alpha'(p)\beta(q) - \alpha(p)\beta'(q)}{p-q} + \frac{\alpha'(p)\beta'(q)}{N} \right) \delta'(x-y) \\ &\quad + pq \left( \frac{\partial_x \alpha(p) \cdot \beta(q) - \alpha(p) \partial_x \beta(q)}{(p-q)^2} \right. \\ &\quad \left. + \frac{\alpha'(p) \partial_x \beta(q) - \alpha(p) \partial_x \beta'(q)}{p-q} + \frac{\alpha'(p) \partial_x \beta'(q)}{N} \right) \delta(x-y), \end{aligned} \quad (3.16)$$

where  $\alpha, \beta \in \{a, \hat{a}\}$ . These Poisson brackets are of hydrodynamic type [16].

Observe that the coefficients of  $\delta'(x-y)$  in (3.15)–(3.16) are exactly the same with the generating functions for the metrics (2.11) and (2.109) on the Frobenius manifold  $\mathcal{M}_{M,N}$ . Therefore, according to the theory of [16, 13], the proposition is proved.  $\square$

One can see that the densities of Hamiltonian functionals  $H_n$  ( $n = 1, 2, \dots, N$ ) and  $\hat{H}_m$  ( $m = 1, 2, \dots, M$ ) are multiple or linear combination of flat coordinates in

$$\{h^j \mid j = 1, \dots, N-1\} \cup \{\hat{h}^k \mid k = 1, \dots, M\} \cup \{t^{-1}\}.$$

By virtue of the bi-hamiltonian recursion relations (3.10), we conclude that the dispersionless Toda lattice hierarchy (3.4)–(3.5) is a subhierarchy of the principal hierarchy associated to the infinite-dimensional Frobenius manifold  $\mathcal{M}_{M,N}$ .

In order to write down the whole principal hierarchy for  $\mathcal{M}_{M,N}$ , one needs to find flat coordinates for the deformed flat metric on this infinite-dimensional Frobenius manifold, which is still open in general. We remark that Carlet and Mertens [9] obtained the principal hierarchy for the infinite-dimensional Frobenius manifold (cf.  $\mathcal{M}_{1,1}$ ) constructed in [7].

**Remark 3.3.** *In [2] Boyarsky et al. derived a system of WDVV associativity equations satisfied by the logarithm of tau function of the dispersionless Toda lattice hierarchy (see also [3, 4]). It is interesting to compare solutions to their associativity equations and the potential  $\mathcal{F}_{M,N}$  given in the present paper. This might be clarified from an answer to a more general question, that is, as indicated at the end of [7], how to extend the potential  $\mathcal{F}_{M,N}$  to the so called topological solution of the principal hierarchy associated to  $\mathcal{M}_{M,N}$ . We will study it elsewhere.*

#### 4. CONCLUSION

We have obtained a class of infinite-dimensional Frobenius manifolds for the dispersionless Toda lattice hierarchy, which generalizes the construction of [7]. Moreover, these infinite-dimensional manifolds contain finite-dimensional Frobenius submanifolds that were constructed on the orbit space of extended affine Weyl groups of type A in [17]. Our result is consistent with the relation between the Toda lattice and the extended bigraded Toda [6, 18] hierarchies as well as their bi-hamiltonian structures [24].

The analogy between the Frobenius manifolds underlying the Toda lattice and the two-component BKP hierarchies is clear. As revised in [arXiv:1103.4048], the factor  $\log(z_2 - z_1)$  in the potential  $\mathcal{F}_{m,n}$  in [25] should be more rigorously replaced by  $\log\left(\frac{z_2 - z_1}{z_2}\right)$  (see (1.19) and (2.63) therein; such a replacement does not change the result due to the residue of an even function is zero). It shows that the approach in [7, 25] and the present paper is indeed efficient for constructing infinite-dimensional Frobenius manifold related to two-component difference or differential integrable hierarchies. What is more, in such constructions, an infinite-dimensional Frobenius manifold can be naturally considered as some extension of a Frobenius manifold of finite dimension. We hope that this observation provides a hint to enlarge the family of infinite-dimensional Frobenius manifold, as well as to understand their relation with the Frobenius manifolds

given in [19, 20]. We plan to study these in separate publications.

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